1. Evaluate the following integrals.

a. $\displaystyle\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{6}{\sqrt{1-x^2}}\,dx$ b. $\displaystyle\int e^{-3x}\cos 2x\,dx$ c. $\displaystyle\int_1^{\sqrt{e}} \frac{dx}{x(\log x - 1)}$

d. $\displaystyle\int \frac{e^{3x}}{9+e^{2x}}\,dx$ e. $\displaystyle\int \sqrt{\tan x}\,\sec^4 x\,dx$

f. $\displaystyle\int \frac{x^2+3x+1}{(2x^2+1)(x-3)}\,dx$ g. $\displaystyle\int \frac{\sqrt{x^2+9}}{x^2}\,dx$

2. Evaluate the following limits.

a. $\displaystyle\lim_{x\to\pi} \frac{1+\cos x}{(x-\pi)^2}$ b. $\displaystyle\lim_{x\to\infty} \left(1+\frac{2}{x}+\frac{1}{x^2}\right)^x$

3. Evaluate the following improper integrals.

a. $\displaystyle\int_4^{\infty} \frac{2}{x^2-2x}\,dx$ b. $\displaystyle\int_{-1}^{1} \sqrt{1+x^{-2/3}}\,dx$

4. Find the value of a such that the line $x = a$ divides the region bounded by the graph of $y = e^x$ and the x-axis on $[\,0, \log 5\,]$ into two regions of equal area.

5. Let $\mathscr{R}$ denote the region bounded by the graphs of

$$y = x + 16/x \qquad \text{and} \qquad y = 10.$$

Set up, but do not evaluate, an integral which represents the volume obtained by revolving $\mathscr{R}$ about a. the x-axis, and b. the line with equation $x = 10$.

6. Find the length of the graph of $y = \frac{2}{3}(x^2+1)^{3/2}$ from $x = 0$ to $x = 3$.

7. Find the explicit solution of the differential equation

$$y' = xe^{2x^2+y},$$

with $y(0) = 0$.

8. Determine whether each sequence converges or diverges. If a sequence converges, find its limit. Justify your answers.

a. $\displaystyle\left\{(-1)^n \frac{2^n}{2^n+n^2}\right\}$ b. $\displaystyle\left\{\frac{5^n}{n!}\right\}$

9. For each of the statements below, show that it is always true or always false, or give an examples to show that it may be true and may be false.

a. If the sequence $\{a_n\}$ converges then the series $\sum a_n$ converges.

b. If the series $\sum a_n$ converges then the sequence $\{a_n\}$ converges.

c. If $\lim\{a_n\sqrt{n}\} = 3$ then $\sum\{1-\cos(a_n)\}$ converges.

d. If $\lim\{a_n\sqrt{n}\} = 3$ then $\sum\{a_n - \sin(a_n)\}$ converges.

10. Find the sum of the series

$$\sum_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n+1}\right)^{n+1}\right\}.$$

11. Determine whether each series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \log\frac{2n^2+3}{n^2+5}$ b. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n}}$

12. Determine whether each of the following series is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{4n-1}{25n+5}\right)^{n/2}$ b. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\arctan n}{\sqrt{n^2+1}}$

13. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(2x-1)^n.$$

14. Find the Taylor series of $\sqrt{x}$ centred at 1. What is its interval of convergence?

1. a. By a standard integral formula,

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{6}{\sqrt{1-x^2}}\,dx = 6\arcsin x\,\Big|_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} = 6\big(\tfrac{1}{3}\pi - \tfrac{1}{6}\pi\big) = \pi.$$

b. Repeated partial integration (integrating the trigonometric factor and differentiating the exponential factor) gives

$$\int e^{-3x}\cos 2x\,dx = \tfrac{1}{2}e^{-3x}\sin 2x - \tfrac{3}{4}e^{-3x}\cos 2x - \tfrac{9}{4}\int e^{-3x}\cos 2x\,dx,$$

and so

$$\int e^{-3x}\cos 2x\,dx = \tfrac{1}{13}e^{-3x}(2\sin 2x - 3\cos 2x) + C.$$

c. Integrating by inspection yields

$$\int_1^{\sqrt{e}} \frac{dx}{x(\log x - 1)} = \log|\log x - 1|\,\Big|_1^{\sqrt{e}} = \log\tfrac{1}{2} = -\log 2.$$

d. Changing the variable of integration to $t = e^x$ (so $dt = e^x\,dx$) yields

$$\begin{aligned}\int \frac{e^{3x}}{9+e^{2x}}\,dx = \int \frac{t^2}{9+t^2}\,dt &= \int\left\{1 - \frac{9}{9+t^2}\right\}dt\\ &= t - 3\arctan\tfrac{1}{3}t + C\\ &= e^x - 3\arctan\tfrac{1}{3}e^x + C.\end{aligned}$$

e. If $t = \tan x$ then $dt = \sec^2 x\,dx$, $\sec^2 x = t^2+1$, and so

$$\begin{aligned}\int \sqrt{\tan x}\,\sec^4 x\,dx = \int t^{1/2}(t^2+1)\,dt &= \int (t^{5/2}+t^{1/2})\,dt\\ &= \tfrac{2}{7}(\tan x)^{7/2} + \tfrac{2}{3}(\tan x)^{3/2} + C.\end{aligned}$$

f. Resolving the integrand into partial fractions (the first coefficient is found by covering and evaluating, and the remaining two by comparing, *e.g.*, quadratic and constant terms) and integrating gives

$$\begin{aligned}\int \frac{x^2+3x+1}{(2x^2+1)(x-3)}\,dx &= \int\left\{\frac{1}{x-3} - \frac{x}{2x^2+1}\right\}dx\\ &= \log|x-3| - \tfrac{1}{4}\log(2x^2+1) + C,\end{aligned}$$

where the second term of the last line is obtained by implicitly changing the variable of integration to $t = 2x^2+1$.

g. Partial integration (integrating the power and differentiating the radical) and a standard integral formula gives

$$\begin{aligned}\int \frac{\sqrt{x^2+9}}{x^2}\,dx &= -\frac{\sqrt{x^2+9}}{x} + \int \frac{dx}{\sqrt{x^2+9}}\\ &= -\frac{\sqrt{x^2+9}}{x} + \ln\big|x + \sqrt{x^2+9}\,\big| + C.\end{aligned}$$

2. a. Expressing the limit in terms of $t = \pi - x$ and expanding about the origin gives

$$\lim_{x\to\pi}\frac{1+\cos x}{(x-\pi)^2} = \lim_{t\to 0}\frac{1-\cos t}{t^2} = \lim_{t\to 0}\big\{\tfrac{1}{2} - \tfrac{1}{24}t^2 + \cdots\big\} = \tfrac{1}{2}.$$

b. Factorizing the base and applying a standard limit formula gives

$$\lim_{x\to\infty}\left(1 + \frac{2}{x} + \frac{1}{x^2}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = e^2.$$

3. a. Since

$$\int \frac{2}{x^2-2x}\,dx = \int \frac{1}{1-2/x}\cdot\frac{2}{x^2}\,dx = \log\big|1-2/x\big| + C$$

(by implicitly changing the variable of integration to $t = 1-2/x$), one has

$$\int_4^\infty \frac{2}{x^2-2x}\,dx = \lim_{\alpha\to\infty}\log\big|1-2/x\big|\,\Big|_4^\alpha = -\log\tfrac{1}{2} = \log 2.$$

b. Let $t = x^{1/3}\sqrt{1+x^{-2/3}}$; then $t^2 = x^{2/3}+1$, so $2t\,dt = \frac{2}{3}x^{-1/3}\,dx$ and hence $3t^2\,dt = \sqrt{1+x^{-2/3}}\,dx$. If $x = 1$ then $t = \sqrt{2}$, and as $x \to 0^+$,

$$t \to \lim_{x\to 0^+} x^{1/3}\sqrt{1+x^{-2/3}} = \lim_{x\to 0^+}\sqrt{x^{2/3}+1} = 1.$$

Since the integrand is an even function, it follows that

$$\begin{aligned}\int_{-1}^1 \sqrt{1+x^{-2/3}}\,dx = 2\int_0^1 \sqrt{1+x^{-2/3}}\,dx &= 2\int_1^{\sqrt{2}} 3t^2\,dt = 2t^3\Big|_1^{\sqrt{2}}\\ &= 4\sqrt{2} - 2.\end{aligned}$$

4. It is required to find a such that $0 < a < \log 5$, and

$$e^a - 1 = \int_0^a e^x\,dx = \frac{1}{2}\int_0^{\log 5} e^x\,dx = \tfrac{1}{2}(5-1) = 2,$$

i.e., $e^a = 3$, and so $a = \log 3$.

5. Since $x + 16/x - 10 = (x-2)(x-8)/x$ is zero if $x = 2, 8$ and negative if $2 < x < 8$, the region $\mathscr{R}$ is bounded above by the horizontal line and below by the curve over the interval $[\,2, 8\,]$ on the x-axis.

a. Cross sections perpendicular to the x-axis of the solid obtained by revolving $\mathscr{R}$ about the x-axis are annuli of outer radius 10 and inner radius $x + 16/x$, for $2 \leqslant x \leqslant 8$, so the volume of this solid is equal to

$$\pi\int_2^8 \big\{100 - (x+16/x)^2\big\}\,dx.$$

b. The solid obtained by revolving $\mathscr{R}$ about the vertical line defined by $x = 10$ can be decomposed into cylindrical shells of radius $10-x$ and height $10-(x+16/x)$, for $2 \leqslant x \leqslant 8$, so its volume is equal to

$$2\pi\int_2^8 (10-x)(10-x-16/x)\,dx.$$

6. If $y = \frac{2}{3}(x^2+1)^{3/2}$ then

$$\begin{aligned}\left(\frac{ds}{dx}\right)^2 &= 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \big(2x(x^2+1)^{1/2}\big)^2\\ &= 1 + 4x^2 + 4x^4\\ &= (1+2x^2)^2,\end{aligned}$$

and so the length of the graph of $y = \frac{2}{3}(x^2+1)^{3/2}$ over $[\,0, 3\,]$ is equal to

$$\int_0^3 (2x^2+1)\,dx = \big\{\tfrac{2}{3}x^3 + x\big\}\,\Big|_0^3 = 21.$$

7. Separating variables and integrating gives

$$\int e^{-y}\,dy = \int xe^{2x^2}\,dx, \qquad \text{or} \qquad e^{-y} = \tfrac{1}{4}(C - e^{2x^2}),$$

and if $y(0) = 0$ then $C = 5$. Therefore, the solution of the initial value problem in question is given by

$$e^{-y} = \tfrac{1}{4}(5 - e^{2x^2}), \qquad \text{or} \qquad y = -\log\big(\tfrac{1}{4}(5-e^{2x^2})\big),$$

provided $-\sqrt{\log\sqrt{5}} < x < \sqrt{\log\sqrt{5}}$.

8. a. Since

$$\lim\frac{n^2}{2^n} = \lim\frac{2}{2^n(\log 2)^2} = 0,$$

by two applications of l'Hôpital's Rule, it follows that

$$\lim\frac{2^n}{2^n+n^2} = 1,$$

and so the sequence $\big\{(-1)^n 2^n/(2^n+n^2)\big\}$ is (oscillating and) divergent.

b. Since

$$0 < \frac{5^n}{n!} < \frac{5^5}{24n},$$

if $n > 6$, it follows that $\lim\{5^n/n!\} = 0$.

Note: One could also apply the Ratio Test ($\varrho = 0$) to conclude that the series $\sum\{5^n/n!\}$ converges, which implies that the sequence $\{5^n/n!\}$ converges to zero by the Vanishing Criterion.

9. a. If $a_n = 1$ for $n \geqslant 1$ then $\lim a_n = 1$ but the series $\sum a_n$ is divergent by the Vanishing Criterion. On the other hand, if $a_n = n^{-2}$ then $\lim a_n = 0$ and $\sum a_n$ is a convergent p-series ($p = 2 > 1$). So the statement in question may be true or may be false.

b. If the series $\sum a_n$ converges then the sequence $\{a_n\}$ converges to zero by the Vanishing Criterion. (Because $\lim a_n = \lim\{s_n - s_{n-1}\} = s - s = 0$, where s is the sum of the series $\sum a_n$ and s_n is the sum of its first n terms.) So the statement in question is always true.

c. If $\lim\{a_n\sqrt{n}\} = 3$ then there is a natural number N such that $2 < a_n < 4$ if $n > N$ In particular, the sequence $\{a_n\}$ is eventually positive. Also, basic properties of limits give $\lim a_n = \lim\{n^{-1/2}\}\lim\{a_n\sqrt{n}\} = 0 \cdot 3 = 0$. Now

$$\begin{aligned}\lim \frac{1-\cos(a_n)}{n^{-1}} &= \lim\left\{\frac{1-\cos(a_n)}{a_n^2}\cdot(a_n\sqrt{n})^2\right\}\\ &= 9\lim_{t\to 0^+}\frac{1-\cos t}{t^2}\\ &= \tfrac{9}{2},\end{aligned}$$

where $t = a_n$, by two applications of l'Hôpital's Rule (or using the Maclaurin expansion of $\cos t$), so $\sum\{1-\cos(a_n)\}$ diverges with the harmonic series by the Limit Comparison Test. Therefore, the statement in question is never true.

d. As in Part c, the hypotheses imply that $\{a_n\}$ is eventually positive and that $\lim a_n = 0$. Also,

$$\begin{aligned}\lim \frac{a_n-\sin(a_n)}{n^{-3/2}} &= \lim\left\{\frac{a_n-\sin(a_n)}{a_n^3}\cdot(a_n\sqrt{n})^3\right\}\\ &= 27\lim_{t\to 0^+}\frac{t-\sin t}{t^3}\\ &= \tfrac{9}{2},\end{aligned}$$

where $t = a_n$ by three applications of l'Hôpital's Rule (or using the Maclaurin expansion of $\sin t$), so $\sum\{a_n - \sin(a_n)\}$ converges with the p-series $\sum n^{-3/2}$ ($p = \frac{3}{2} > 1$) by the Limit Comparison Test. Therefore, the statement in question is always true.

10. If $b_n = \left(1+\frac{1}{n}\right)^n$ then the series in question is equal to

$$\sum_{n=1}^{\infty}\{b_n - b_{n+1}\} = b_1 - \lim b_{n+1} = 2 - e,$$

by a standard limit formula.

11. a. Since

$$\lim \log\frac{2n^2+3}{n^2+5} = \log 2 \neq 0,$$

the series in question diverges by the Vanishing Criterion.

b. If $a_n = ne^{-n^2}$ and $b_n = 1/n^3$ then

$$\lim\frac{a_n}{b_n} = \lim\frac{n^4}{e^{n^2}} = \lim_{t\to\infty}\frac{t^2}{e^t} = \lim_{t\to\infty}\frac{2}{e^t} = 0,$$

where $t = n^2$, by two applications of l'Hôpital's Rule. Since $\sum b_n$ is a convergent p-series ($p = 3 > 1$), the series $\sum a_n$ converges by the Limit Comparison Test.

c. Since

$$0 < \left|\frac{\sin n}{n\sqrt{n}}\right| < \frac{1}{n^{3/2}}$$

and $\sum n^{-3/2}$ is a convergent p-series ($p = \frac{3}{2} > 1$), the Comparison Test implies that the series

$$\sum\frac{\sin n}{n\sqrt{n}}$$

is convergent.

12. a. If

$$a_n = \left(\frac{4n-1}{25n+5}\right)^{n/2}$$

then $\lim\sqrt[n]{a_n} = \sqrt{\frac{4}{25}} = \frac{2}{5} < 1$, and so the series $\sum(-1)^n a_n$ is absolutely convergent by the Root Test.

b. If

$$a_n = \frac{\arctan n}{\sqrt{n^2+1}}$$

then

$$a_n \geqslant \frac{\pi/4}{\sqrt{n^2+1}} \geqslant \frac{\pi/4}{\sqrt{n^2+n^2}} = \frac{\pi}{4}\cdot\frac{1}{\sqrt{2n^2}} = \frac{\pi}{2\sqrt{2}}\cdot\frac{1}{n} > 0,$$

if $n \geqslant 1$, so the Comparison Test implies that the series $\sum a_n$ diverges with the harmonic series. On the other hand, a_n is positive if $n \geqslant 1$,

$$\frac{d}{dx}\left\{\frac{\arctan x}{\sqrt{x^2+1}}\right\} = \frac{1-x\arctan x}{(x^2+1)^{3/2}} < 0,$$

e.g. if $x > \sqrt{3}$ (since $\frac{1}{3}\pi\sqrt{3} > 1$ and $x \rightsquigarrow x\arctan x$ is a product of increasing positive functions), so $\{a_n\}_{n\geqslant 2}$ is decreasing. Finally, if $n \geqslant 1$ then

$$0 < a_n < \frac{\pi/2}{\sqrt{n^2+1}} \qquad\text{and}\qquad \lim\frac{\pi/2}{\sqrt{n^2+1}} = 0,$$

so the Squeeze Theorem. implies that $\lim a_n = 0$. Hence, $\sum(-1)^n a_n$ converges by the Alternating Series Test. Therefore, the series $\sum(-1)^n a_n$ is conditionally convergent.

13. If

$$\alpha_n = \frac{(n!)^2}{(2n)!}(2x-1)^n,$$

then

$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim\frac{n+1}{2(2n+1)}|2x-1| = \tfrac{1}{4}|2x-1|$$

so by the Ratio Test the series $\sum\alpha_n$ is absolutely convergent if $\frac{1}{4}|2x-1| < 1$, *i.e.*, $-\frac{3}{2} < x < \frac{5}{2}$, and divergent if $x < -\frac{3}{2}$ or $x > \frac{5}{2}$. At the endpoints of the interval of convergence

$$\alpha_n = (\pm 4)^n\frac{(n!)^2}{(2n)!},$$

and so

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \frac{4(n+1)}{2(2n+1)} = \frac{n+1}{n+1/2} > 1$$

which implies that $\lim\alpha_n \neq 0$, and so $\sum\alpha_n$ diverges by the Vanishing Criterion. Therefore, the radius of convergence of $\sum\alpha_n$ is 2 and the interval of convergence of $\sum\alpha_n$ is $(-\frac{3}{2}, \frac{5}{2})$.

Note: The divergence of $\sum\alpha_n$ at the endpoints of its interval of convergence could have been checked easily using the Gauß Test, but that is not necessary in this case.

14. One has

$$\begin{aligned}\sqrt{x} &= \sqrt{1+(x-1)}\\ &= 1 + \sum_{k=1}^{\infty}\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{3}{2}-k)}{k!}(x-1)^k\\ &= 1 + \sum_{k=1}^{\infty}(-1)^k\frac{(-1)\cdot 1\cdot 3\cdots(2k-3)}{2\cdot 4\cdot 6\cdots(2k)}(x-1)^k\\ &= \sum_{k=0}^{\infty}\frac{(-1)^{k-1}(2k)!}{2^{2k}(k!)^2(2k-1)}(x-1)^k,\end{aligned}$$

which is valid at least for $-1 < x-1 < 1$, *i.e.*, $0 < x < 2$. Applying the Gauß Test at the endpoints gives

$$\left|\frac{u_{k+1}}{u_k}\right| = \frac{2k-1}{2k+2} = \frac{k-1/2}{k+1}$$

and so $\sum u_k$ is absolutely convergent since $-\frac{1}{2} - 1 = -\frac{3}{2} < 1$. Therefore the interval of convergence of the Taylor series of $\sqrt{x}$ centred at 1 is $[0, 2]$.