1. Given $f(x) = \operatorname{arcsec}\sqrt{x^2+1}$, find $f'(x)$ and simplify your answer.

2. Evaluate each of the following integrals.

a. $\displaystyle\int x\sqrt[3]{x-1}\,dx$ b. $\displaystyle\int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2-4}}{x}\,dx$

c. $\displaystyle\int t^2 \arcsin t\,dt$ d. $\displaystyle\int (a+\tan x)^2\,dx$

e. $\displaystyle\int \sqrt{z^2+4z+5}\,dz$ f. $\displaystyle\int \frac{x^2+2x+5}{x^2(x^2+1)}\,dx$

3. Evaluate the improper integral or explain why it diverges.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4+x^2}$ b. $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$

4. Find all values of p for which the following improper integral is convergent.

$$\mathscr{I}_p = \int_0^\infty \cos(x^p)\,dx$$

5. Evaluate each of the following limits.

a. $\displaystyle\lim_{\vartheta\to 0} \frac{2\vartheta - \sin 2\vartheta}{\vartheta - \sin\vartheta}$ b. $\displaystyle\lim_{x\to 0^+} (1+\sin 2x)^{1/x}$

6. Find the area of the region bounded by the graphs of $y = 4/x$ and $y = 5 - x$.

7. Set-up an integral which represents the volume of the solid obtained by
a. revolving the region enclosed by the graphs of $y = x$, $y = \frac{1}{3}x$ and $x = 2$, about the y-axis.
b. revolving the region enclosed by the graphs of $y = x$, $y = \frac{1}{3}x$ and $y = -1$, about the line defined by $x = 3$.

8. Find the length of the curve defined by $y = 2x^{3/2} + 1$, for $0 \leqslant x \leqslant \frac{1}{3}$.

9. A tank contains 50 kg of salt dissolved in 1500 L of water. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. a. How much salt is in the tank after t minutes? b. How much salt is in the tank after 150 minutes?

10. The sequence $\{a_n\}$ is defined by

$$a_n = \frac{3n^2 + \sin n}{5n^2 + n},$$

for $n \geqslant 1$.
a. Does $\{a_n\}$ converge? Justify your answer.
b. Does

$$\sum_{n=1}^\infty a_n$$

converge? Justify your answer.

11. Determine whether each of the following series is convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=0}^\infty \frac{n}{\sqrt{n^3+1}}$ b. $\displaystyle\sum_{k=1}^\infty \left\{1 - \cos\left(\frac{\pi}{2k}\right)\right\}$

12. Determine whether each of the following series is absolutely convergent, conditionally convergent or divergent. Justify your conclusions.

a. $\displaystyle\sum_{n=1}^\infty (-1)^n \frac{\log(n+1)}{n+1}$ b. $\displaystyle\sum_{n=1}^\infty (-1)^{n-1} \frac{n^2}{n!}$

13. Determine whether each of the following series converges or diverges. If a series converges, find its sum.

a. $\displaystyle\sum_{m=0}^\infty \frac{2^{m+1} + 7^m}{3^m}$ b. $\displaystyle\sum_{n=2}^\infty \frac{1}{n^2-1}$

14. Find the Taylor series centered at 1 of

$$f(x) = \log(1+x).$$

15. Determine the radius and interval of convergence of the series

$$\sum_{n=1}^\infty \frac{n(x-2)^n}{n^2+1}.$$

1. If $f(x) = \operatorname{arcsec}\sqrt{x^2+1}$, then by the Chain Rule

$$
\begin{aligned}
f'(x) &= \frac{1}{\sqrt{x^2+1}\sqrt{(x^2+1)-1}} \cdot \frac{x}{\sqrt{x^2+1}} = \frac{x}{|x|(x^2+1)} \\
&= \begin{cases} \dfrac{1}{x^2+1} & \text{if } x > 0, \text{ and} \\ \dfrac{-1}{x^2+1} & \text{if } x < 0. \end{cases}
\end{aligned}
$$

2. a. Repeated partial integration gives

$$
\begin{aligned}
\int x\sqrt[3]{x-1}\,dx &= \tfrac{3}{4}x(x-1)^{4/3} - \tfrac{3}{4}\cdot\tfrac{3}{7}(x-1)^{7/3} + C \\
&= \tfrac{3}{28}(4x+3)\sqrt[3]{(x-1)^4} + C.
\end{aligned}
$$

b. If $5x = 2\sec\vartheta$ then $dx = \frac{2}{5}\sec\vartheta\tan\vartheta\,d\vartheta$ and $\sqrt{25x^2-4} = 2\tan\vartheta$. Also, $\vartheta = 0$ when $x = \frac{2}{5}$, and $\vartheta = \frac{1}{3}\pi$ when $x = \frac{4}{5}$. Therefore,

$$
\begin{aligned}
\int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2-4}}{x}\,dx &= \int_0^{\frac{1}{3}\pi} \frac{(2\tan\vartheta)(\frac{2}{5}\sec\vartheta\tan\vartheta)}{\frac{2}{5}\sec\vartheta}\,d\vartheta \\
&= 2\int_0^{\frac{1}{3}\pi} \tan^2\vartheta\,d\vartheta = 2\int_0^{\frac{1}{3}\pi} (\sec^2\vartheta - 1)\,d\vartheta \\
&= 2(\tan\vartheta - \vartheta)\Big|_0^{\frac{1}{3}\pi} \\
&= 2\sqrt{3} - \tfrac{2}{3}\pi.
\end{aligned}
$$

c. Partial integration gives

$$
\int t^2 \arcsin t\,dt = \tfrac{1}{3}t^3\arcsin t - \tfrac{1}{3}\int \frac{t^3}{\sqrt{1-t^2}}\,dt.
$$

Adding and subtracting t and integrating by inspection then yields

$$
\begin{aligned}
\int \frac{t - t^3 - t}{\sqrt{1-t^2}}\,dt &= \int t\sqrt{1-t^2}\,dt - \int \frac{t}{\sqrt{1-t^2}}\,dt \\
&= -\tfrac{1}{3}(1-t^2)^{3/2} + \sqrt{1-t^2} + C \\
&= \tfrac{1}{3}(t^2+2)\sqrt{1-t^2} + C,
\end{aligned}
$$

and therefore,

$$
\int t^2\arcsin t\,dt = \tfrac{1}{3}t^3\arcsin t + \tfrac{1}{9}(t^2+2)\sqrt{1-t^2} + C.
$$

d. Expanding, using a Pythagorean identity and integrating term-by-term gives

$$
\begin{aligned}
\int (a+\tan x)^2\,dx &= \int (a^2 + 2a\tan x + \tan^2 x)\,dx \\
&= \int (a^2 - 1 + + \sec^2 x + 2a\tan x)\,dx \\
&= (a^2-1)x + \tan x - 2a\log|\cos x| + C.
\end{aligned}
$$

e. Completing the square gives $\sqrt{z^2+4z+5} = \sqrt{(z+2)^2+1}$. Next, if $z+2 = \tan\vartheta$ then $dz = \sec^2\vartheta\,d\vartheta$ and $\sqrt{z^2+4z+5} = \sec\vartheta$. Expressing the integral in terms of ϑ, integrating by parts (integrating $\sec^2\vartheta$ and differentiating $\sec\vartheta$), and applying a Pythagorean identity gives

$$
\begin{aligned}
\int \sqrt{z^2+4z+5}\,dz &= \int \sec^3\vartheta\,d\vartheta = \sec\vartheta\tan\vartheta - \int \sec\vartheta\tan^2\vartheta\,d\vartheta \\
&= \sec\vartheta\tan\vartheta - \int \sec\vartheta(\sec^2\vartheta - 1)\,d\vartheta \\
&= \sec\vartheta\tan\vartheta + \int \sec\vartheta\,d\vartheta - \int \sec^3\,\vartheta,
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\int \sqrt{z^2+4z+5}\,dz &= \tfrac{1}{2}\sec\vartheta\tan\vartheta + \tfrac{1}{2}\ln|\sec\vartheta + \tan\vartheta| + C \\
&= \tfrac{1}{2}(z+2)\sqrt{z^2+4z+5} + \tfrac{1}{2}\ln\left|z+2+\sqrt{z^2+4z+5}\right| + C.
\end{aligned}
$$

f. The resolution into partial fractions of the integrand has the form

$$
\frac{x^2+2x+5}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1},
$$

and $B = 5$ is obtained by covering up x^2. Clearing denominators yields

$$
x^2+2x+5 = Ax(x^2+1) + 5(x^2+1) + (Cx+D)x^2.
$$

Comparing the coefficients of x^3 gives $A + C = 0$, and so $C = -A$. Comparing the coefficients of x^2 gives $1 = 5 + D$, and so $D = -4$. Comparing the coefficients of x gives $A = 2$, which implies that $C = -2$. Therefore, the integral in question is equal to

$$
\int \left\{ \frac{2}{x} + \frac{5}{x^2} + \frac{-2x-4}{x^2+1} \right\} dx = -\frac{5}{x} + \log\frac{x^2}{x^2+1} - 4\arctan x + C.
$$

3. a. By symmetry and an elementary integral formula one has

$$
\int_{-\infty}^{\infty} \frac{dx}{x^2+4} = 2\int_0^{\infty} \frac{dx}{x^2+4} = \lim_{\alpha\to\infty} \arctan\left(\tfrac{1}{2}x\right)\Big|_0^{\alpha} = \tfrac{1}{2}\pi.
$$

b. Since

$$
\int_0^1 \frac{dx}{(x-1)^2} = \lim_{\alpha\to1^-} \frac{-1}{x-1}\Big|_0^{\alpha} = \infty
$$

and

$$
\int_1^2 \frac{dx}{(x-1)^2} = \lim_{\beta\to1^+} \frac{-1}{x-1}\Big|_{\beta}^{2} = \infty,
$$

the integral in question diverges to ∞.

4. First observe that if $p \leqslant 0$ and x is sufficiently large ($x > 0$ if $p = 0$, and $x > (\pi/3)^{1/p}$ if $p < 0$) then $0 < x^p < \frac{1}{3}\pi$, and so $0 < \frac{1}{2} < \cos(x^p)$. Since the integral of $\frac{1}{2}$ on $[\,0,\infty\,)$ diverges to ∞, it follows that $\mathscr{I}_p$ diverges to ∞ if $p \leqslant 0$. Next, if $0 < p \leqslant 1$, then $|x^{p-1}\cos(x^p)| \leqslant |\cos(x^p)|$, provided $x \geqslant 1$; so

$$
\int_{\sqrt[p]{2n\pi - \frac{1}{2}\pi}}^{\sqrt[p]{2n\pi + \frac{1}{2}\pi}} \cos(x^p)\,dx \geqslant \int_{\sqrt[p]{2n\pi - \frac{1}{2}\pi}}^{\sqrt[p]{2n\pi + \frac{1}{2}\pi}} x^{p-1}\cos(x^p)\,dx = 2/p,
$$

and

$$
\int_{\sqrt[p]{2n\pi + \frac{1}{2}\pi}}^{\sqrt[p]{2n\pi + \frac{3}{2}\pi}} \cos(x^p)\,dx \leqslant \int_{\sqrt[p]{2n\pi + \frac{1}{2}\pi}}^{\sqrt[p]{2n\pi + \frac{3}{2}\pi}} x^{p-1}\cos(x^p)\,dx = -2/p.
$$

Therefore, $\mathscr{I}_p$ diverges and oscillates if $0 < p \leqslant 1$. Finally, if $p > 1$ then

$$
\mathscr{I}_p = \int_0^{\sqrt[p]{\pi}} \cos(x^p)\,dx + \int_{\sqrt[p]{\pi}}^{\infty} \cos(x^p)\,dx,
$$

where the first term is a proper integral, and

$$
\begin{aligned}
\int_{\sqrt[p]{\pi}}^{\infty} \cos(x^p)\,dx &= \int_{\sqrt[p]{\pi}}^{\infty} \frac{x^{p-1}\cos(x^p)}{x^{p-1}}\,dx \\
&= \lim_{t\to\infty} \frac{\sin(x^p)}{px^{p-1}}\Big|_{\sqrt[p]{\pi}}^{t} + \frac{p-1}{p}\int_{\sqrt[p]{\pi}}^{\infty} \frac{\sin(x^p)}{x^p}\,dx \\
&= \frac{p-1}{p}\int_{\sqrt[p]{\pi}}^{\infty} \frac{\sin(x^p)}{x^p}\,dx
\end{aligned}
$$

by the Squeeze Theorem. Now

$$
\left|\frac{\sin(x^p)}{x^p}\right| \leqslant \frac{1}{x^p}
$$

and

$$
\int_1^{\infty} \frac{dx}{x^p}
$$

is a convergent improper integral since $p > 1$, so the comparison principle implies that

$$
\int_{\sqrt[p]{\pi}}^{\infty} \frac{\sin(x^p)}{x^p}\,dx
$$

converges. Therefore, $\mathscr{I}_p$ converges if, and only if, $p > 1$.

5. a. Using the Maclaurin expansion of the sine function gives

$$
\begin{aligned}
\lim_{\vartheta\to0} \frac{2\vartheta - \sin 2\vartheta}{\vartheta - \sin\vartheta} &= \lim_{\vartheta\to0} \frac{\frac{1}{6}(2\vartheta)^3 - \frac{1}{120}(2\vartheta)^5 + \cdots}{\frac{1}{6}\vartheta^3 - \frac{1}{120}\vartheta^5 + \cdots} \\
&= 8\lim_{\vartheta\to0} \frac{\frac{1}{6} - \frac{1}{120}(2\vartheta)^2 + \cdots}{\frac{1}{6} - \frac{1}{120}\vartheta^2 + \cdots} \\
&= 8.
\end{aligned}
$$

b. Since
$$\lim_{x\to 0^+}\frac{\log(1+\sin 2x)}{x}=\lim_{x\to 0^+}\frac{2\cos 2x}{1+\sin 2x}=2,$$
it follows that
$$\lim_{x\to 0^+}(1+\sin 2x)^{1/x}=e^2.$$

6. The curves meet where $4/x=5-x$, *i.e.*, $0=x^2-5x+4=(x-1)(x-4)$. If $1<x<4$ then $(x-1)(x-4)<0$, so the curve is below the line. Therefore, the area of the region enclosed by the graphs of the given equations is equal to
$$\int_1^4(-x+5-4/x)\,dx=\left\{-\tfrac{1}{2}x^2+5x-4\log x\right\}\Big|_1^4=\tfrac{15}{2}-8\log 2.$$

7. a. The solid in question can be decomposed into concentric cylinders of radius x and height $x-\frac{1}{3}x=\frac{2}{3}x$, for $0\leqslant x\leqslant 2$, so its volume is equal to
$$2\pi\int_0^2\tfrac{2}{3}x^2\,dx.$$
b. The solid in question can be decomposed into annuli of inner radius $3-y$ and outer radius $3-3y$, for $-1\leqslant y\leqslant 0$, so its volume is equal to
$$\pi\int_{-1}^0\left\{(3-3y)^2-(3-y)^2\right\}dy.$$

8. If $y=2x^{3/2}+1$ then $dy/dx=3\sqrt{x}$, and so
$$\left(\frac{ds}{dx}\right)^2=1+(3\sqrt{x})^2=1+9x.$$
Therefore, the length of the curve defined by $y=2x^{3/2}+1$ and $0\leqslant x\leqslant\frac{1}{3}$ is equal to
$$\int_0^{\frac{1}{3}}\sqrt{1+9x}\,dx=\tfrac{2}{27}(1+9x)^{3/2}\Big|_0^{\frac{1}{3}}=\tfrac{14}{27}.$$

9. The amount q of salt (measured in kilograms) in the tank after t minutes satisfies the differential equation
$$\frac{dq}{dt}=-\tfrac{1}{150}q,$$
which (since it represents exponential decay) is given by $q=Ae^{-t/150}$, where $A=q(0)=50$. Therefore, a. $q=50e^{-t/150}$, and b. $q(150)=50/e$ kg.

10. a.
$$\lim a_n=\lim\frac{3+(\sin n)/n^2}{5+1/n}=\tfrac{3}{5},$$
since $\lim\left\{(\sin n)/n^2\right\}=0$ by the Squeeze Theorem.

b. Since $\lim a_n\neq 0$, the series $\sum a_n$ diverges by the Vanishing Criterion.

11. a. Since
$$\frac{n}{\sqrt{n^3+1}}\geqslant\frac{n}{\sqrt{n^3+n^3}}=\tfrac{1}{2}\sqrt{2}\,n^{-1/2}>0,$$
for $n\geqslant 1$, and $\sum n^{-1/2}$ is a divergent p-series ($p=\frac{1}{2}\leqslant 1$), the series
$$\sum_{n=0}^{\infty}\frac{n}{\sqrt{n^3+1}}$$
diverges by the Comparison Test.

b. Let
$$a_k=1-\cos\left(\frac{\pi}{2k}\right)\qquad\text{and}\qquad b_k=k^{-2};$$
then
$$\lim\frac{a_k}{b_k}=\tfrac{1}{4}\pi^2\lim_{t\to 0^+}\frac{1-\cos t}{t^2}=\tfrac{1}{8}\pi^2,$$
where $t=\frac{1}{2}\pi k^{-1}$, by two applications of l'Hôpital's Rule (or using the Maclaurin series of the cosine function). Since $\sum b_k$ is a convergent p-series ($p=2>1$), the Limit Comparison Test implies that $\sum a_k$ is convergent.

12. a. This series is a (non-zero multiple of a) conditionally convergent alternating logarithmic p-series ($p=-1\leqslant 1$). Pedantically, observe that the sequence of terms of the series is $\left\{(-1)^{n-1}a_n\right\}_{n\geqslant 2}$, where $a_n=(\log n)/n$. Since $a_n>n^{-1}$ if $n\geqslant 3$, the series $\sum a_n$ diverges with the harmonic series by the Comparison Test, so $\sum(-1)^{n-1}a_n$ is not absolutely convergent. Next, $a_n>0$ for $n\geqslant 2$,
$$\frac{d}{dx}\left\{\frac{\log x}{x}\right\}=\frac{1-\log x}{x^2}$$
is negative if $x>e$ so $\{a_n\}$ is eventually decreasing, and $\lim a_n=0$ by one application of l'Hôpital's Rule, so $\sum(-1)^{n-1}a_n$ converges by the Alternating Series Test. Therefore, $\sum(-1)^{n-1}a_n$ is conditionally convergent.

b. Let $a_n=n^2/n!$; then
$$\lim\frac{a_{n+1}}{a_n}=\lim\left\{\frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2}\right\}==\lim\left\{\frac{1}{n}\left(1+\frac{1}{n}\right)\right\}=0,$$
so $\sum(-1)^{n-1}a_n$ is absolutely convergent by the Ratio Test.

13. a. Since $a_m=\left(2^{m+1}+7^m\right)/3^m>\left(\frac{7}{3}\right)^m$ for $m\geqslant 0$, and $\sum\left(\frac{7}{3}\right)^m$ is a divergent geometric series ($r=\frac{7}{3}\geqslant 1$), the Comparison Test implies that $\sum a_m$ diverges (to ∞).

b. Since
$$\frac{1}{n^2-1}=\frac{1}{2}\left\{\frac{1}{n-1}-\frac{1}{n+1}\right\}=b_n-b_{n+1},$$
for $n\geqslant 2$, where
$$b_n=\frac{1}{2}\left\{\frac{1}{n-1}+\frac{1}{n}\right\},$$
it follows that
$$\sum_{n=2}^{\infty}\frac{1}{n^2-1}=b_2-\lim b_{n+2}=\tfrac{3}{4}.$$

14. The Maclaurin series and the functional equation of the logarithm give
$$\begin{aligned}\log(1+x)&=\log(2+x-1)=\log 2+\log\left(1+\tfrac{1}{2}(x-1)\right)\\&=\log 2+\sum_{n=1}\frac{(-1)^{n-1}}{2^n n}(x-1)^n.\end{aligned}$$

15. Let
$$u_n=\frac{n(x-2)^n}{n^2+1};$$
then
$$\begin{aligned}\lim\left|\frac{u_{n+1}}{u_n}\right|&=\lim\left|\frac{(n+1)(x-2)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{n(x-2)^n}\right|\\&=\lim\left(1+\frac{1}{n}\right)\frac{1+1/n^2}{(1+1/n)^2+1/n^2}|x-2|\\&=|x-2|,\end{aligned}$$
so by the Ratio Test, $\sum u_n$ converges if $|x-2|<1$, *i.e.*., $1<x<3$, and diverges if $x<1$ or $x>3$. If $x=3$ then
$$u_n=\frac{n}{n^2+1}\geqslant\frac{n}{n^2+n^2}=\tfrac{1}{2}n^{-1}$$
if $n\geqslant 1$, so $\sum u_n$ diverges with the harmonic series by the Comparison Test. If $x=1$ then
$$u_n=(-1)^n a_n,\qquad\text{where}\qquad a_n=\frac{n}{n^2+1}.$$
Now a_n is positive if $n\geqslant 1$, and
$$\lim a_n=\lim\left\{\frac{1}{n}\cdot\frac{1}{1+1/n^2}\right\}=0.$$
Finally,
$$\frac{d}{dx}\left\{\frac{x}{x^2+1}\right\}=\frac{1-x^2}{(x^2+1)^2}$$
is negative if $x>1$, which implies that the sequence $\left\{a_n\right\}_{n\geqslant 1}$ is decreasing, so the Alternating Series Test implies that $\sum(-1)^n a_n$ converges. Therefore, the radius of convergence of $\sum u_n$ is 1 and the interval of convergence of $\sum u_n$ is $[1,3)$.