1. Evaluate each of the following integrals.

a. $\displaystyle\int_{\sqrt{2}}^{2} \frac{3}{x\sqrt{x^2-1}}\,dx$ b. $\displaystyle\int \frac{xe^x}{\sqrt{xe^x - e^x}}\,dx$ c. $\displaystyle\int \frac{\log x}{x^3}\,dx$

d. $\displaystyle\int \sec x \tan^5 x\,dx$ e. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

f. $\displaystyle\int \frac{x-6}{x^3(x-2)}\,dx$ g. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos^3 x}{1+\sin^2 x}\,dx$

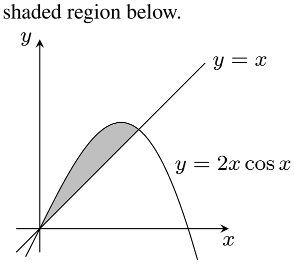
2. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to\infty} \left\{x\left(\arctan 3x - \tfrac{1}{2}\pi\right)\right\}$ b. $\displaystyle\lim_{x\to 0^+} (\sec x + \tan x)^{1/x}$

3. Evaluate each of improper integral or show that it diverges.

a. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ b. $\displaystyle\int_3^\infty \frac{1}{\sqrt[5]{x-2}}\,dx$

4. Find the area of the shaded region below.



5. Let $\mathscr{R}$ denote the region bounded by the graphs of $x = 2\sqrt{y}$, $x = y+2$, $y = 0$ and $y = 4$. Set up, but do not evaluate, an integral which represents the volume of the solid obtained by revolving $\mathscr{R}$ about

a. the x-axis, b. the line defined by $x = -1$.

6. Find the length of the curve defined by $y = \frac{1}{12}x^3 + x^{-1}$, $1 \leqslant x \leqslant 2$.

7. Express y as a function of x if

$$\frac{dy}{dx} = \frac{x}{x^2y+y}, \qquad \text{and} \qquad y = -4 \quad \text{if} \quad x = 0.$$

8. Let $f(x) = \sin\left(\frac{1}{2}x\right)$, and let $a_n = f^{(n)}(0)$, for $n \geqslant 1$.

a. Find the first five terms of the sequence $\{a_n\}$.

b. Find $\lim a_n$ or explain why the sequence $\{a_n\}$ diverges.

c. Find the sum of

$$\sum_{n=1}^{\infty} \frac{(5\pi)^n a_n}{2^n n!},$$

or explain why this series is divergent.

9. Let

$$\sum_{n=1}^{\infty} a_n$$

be the series whose n^{th} partial sum is

$$s_n = \frac{5n}{2n+1}.$$

a. Evaluate $\displaystyle\sum_{n=1}^{\infty} a_n$. b. Find a_3.

10. Determine whether the series converges or diverges. Justify your answer.

a. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{3n}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+7^n}{2^n+9^n}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n\sqrt{n+1}}$

11. Determine whether the series is absolutely convergent, conditionally convergent or divergent. Justify your answer.

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^n (n!)^2}{(2n)!}$ b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$

12. Find the radius and interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2x-3)^n}{3^n\sqrt{2n+3}}.$$

13. Find the Maclaurin series of

$$f(x) = \frac{1}{\sqrt{2x+1}}.$$

What is the interval of convergence of this series?

14. In this problem you are given a convergent series

$$\sum_{n=0}^{\infty} a_n.$$

a. Give an example to show that

$$\sum_{n=1}^{\infty} (a_n)^2$$

may diverge.

b. Prove that if there is a positive integer k such that $a_n \geqslant 0$ for $n \geqslant k$, then

$$\sum_{n=0}^{\infty} (a_n)^2$$

is convergent.

1. a. A basic integral formula yields

$$\int_{\sqrt{2}}^{2} \frac{3}{x\sqrt{x^2-1}}\,dx = 3\operatorname{arcsec} x \,\Big|_{\sqrt{2}}^{2} = \pi - \tfrac{3}{4}\pi = \tfrac{1}{4}\pi.$$

b. If $t = \sqrt{xe^x - e^x}$, then $t^2 = xe^x - e^x$ and $2t\,dt = xe^x\,dx$, and so

$$\int \frac{xe^x}{\sqrt{xe^x - e^x}} = \int 2\,dt = 2t + C = 2\sqrt{xe^x - e^x} + C.$$

c. Partial integration, integrating the power and differentiating the logarithm, gives

$$\int \frac{\log x}{x^3}\,dx = -\frac{\log x}{2x^2} + \int \frac{1}{2x^3}\,dx = -\frac{\log x}{2x^2} - \frac{1}{4x^2} + C.$$

d. If $t = \sec x$ then $dt = \sec x \tan x\,dx$, $\tan^4 x = (t^2-1)^2 = t^4 - 2t^2 + 1$, and so

$$\int \sec x \tan^5 x\,dx = \int (t^4 - 2t^2 + 1)\,dt = \tfrac{1}{5}\sec^5 x - \tfrac{2}{3}\sec^3 x + \sec x + C.$$

e. Partial integation gives

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= \int x \cdot \frac{x}{\sqrt{16-x^2}}\,dx = -x\sqrt{16-x^2} + \int \sqrt{16-x^2}\,dx \\ &= -x\sqrt{16-x^2} + \int \frac{16}{\sqrt{16-x^2}}\,dx - \int \frac{x^2}{\sqrt{16-x^2}}\,dx \\ &= -\tfrac{1}{2}x\sqrt{16-x^2} + 8\arcsin\left(\tfrac{1}{4}x\right) + C.\end{aligned}$$

(A less effective calculation would begin by letting $x = 4\sin\vartheta$.)

f. The resolution into partial fractions of the integrand has the form

$$\frac{x-6}{x^3(x-2)} + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2},$$

and $C = 3$ and $D = -\frac{1}{2}$ are obtained by covering. Clearing denominators gives

$$x - 6 = Ax^2(x-2) + Bx(x-2) + C(x-2) + Dx^3.$$

Comparing the coefficients of x^3 gives $A + D = 0$, so $A = \frac{1}{2}$, and then comparing the coefficients of x gives $-2B + C = 1$, so $B = 1$. Therefore,

$$\begin{aligned}\frac{x-6}{x^3(x-2)} &= \int \left\{ \frac{1}{2x} + \frac{1}{x^2} + \frac{3}{x^3} - \frac{1}{2(x-2)} \right\} dx \\ &= \tfrac{1}{2}\log\left|\frac{x}{x-2}\right| - \frac{1}{x} - \frac{3}{2x^2} + C.\end{aligned}$$

g. If $t = \sin x$ then $dt = \cos x\,dx$ and $\cos^2 x = 1 - t^2$, and so

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \frac{\cos^3 x}{1 + \sin^2 x}\,dx &= \int_0^1 \frac{1-t^2}{1+t^2}\,dt = \int_0^1 \left\{ -1 + \frac{2}{1+t^2} \right\} dt \\ &= \left\{ -t + 2\arctan t \right\} \Big|_0^1 \\ &= \tfrac{1}{2}\pi - 1.\end{aligned}$$

2. a. Revising the expression in the limit and then applying l'Hôpital's Rule gives

$$\begin{aligned}\lim_{x\to\infty} \left\{ x\left(\arctan 3x - \tfrac{1}{2}\pi\right) \right\} &= \lim_{x\to\infty} \frac{\arctan 3x - \frac{1}{2}\pi}{1/x} \\ &= \lim_{x\to\infty} \frac{-3x^2}{1+9x^2} = \lim_{x\to\infty} \frac{-3}{1/x^2 + 9} \\ &= -\tfrac{1}{3}.\end{aligned}$$

b. Since

$$\lim_{x\to 0^+} \frac{\log(\sec x + \tan x)}{x} = \lim_{x\to 0^+} \sec x = 1,$$

by l'Hôpital's Rule (or the definition of the derivatve of a function), it follows that

$$\lim_{x\to 0^+} (\sec x + \tan x)^{1/x} = e.$$

3. a. Integrating by inspection gives

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{\delta\to 0^+} 2e^{\sqrt{x}} \,\Big|_\delta^1 = 2(e-1).$$

b. Integrating by inspection gives

$$\int_3^\infty \frac{1}{\sqrt[5]{x-2}}\,dx = \lim_{t\to\infty} \tfrac{5}{4}(x-2)^{4/5} \,\Big|_3^t = \infty,$$

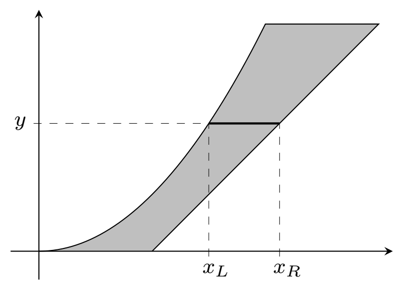
so the integral in question diverges. (The divergence of this integral could have been explained by appeal to a standard scale.)

4. The curves intersect where $2x\cos x = x$, or $x(2\cos x - 1) = 0$, *i.e.*, where $x = 0$ or $\cos x = \frac{1}{2}$. The solutions of the latter equation are $\pm\frac{1}{3}\pi + 2k\pi$, where k is an integer, so the points of intersection in the given sketch of the region are the origin and $\left(\frac{1}{3}\pi, \frac{1}{3}\pi\right)$. The curve defined by $y = 2x\cos x$ is above the line defined by $y = x$ (this is shown in the given sketch, but it can be deduced from the fact that $x(2\cos x - 1)$ is positive if $0 < x < \frac{1}{3}\pi$), so the area of the shaded region is equal to

$$\begin{aligned}\int_0^{\frac{1}{3}\pi} (2x\cos x - x)\,dx &= \left(2x\sin x - \tfrac{1}{2}x^2\right)\Big|_0^{\frac{1}{3}\pi} - \int_0^{\frac{1}{3}\pi} 2\sin x\,dx \\ &= \tfrac{1}{3}\pi\sqrt{3} - \tfrac{1}{18}\pi^2 + 2\cos x \,\Big|_0^{\frac{1}{3}\pi} \\ &= \tfrac{1}{3}\pi\sqrt{3} - \tfrac{1}{18}\pi^2 - 1,\end{aligned}$$

where the first term of the integrand was integrated by parts.

5. Since $2\sqrt{y} < y + 2$ if $0 \leqslant y \leqslant 4$, the region $\mathscr{R}$ is composed of horizontal segments of length $y - 2\sqrt{y} + 2$, for $0 \leqslant y \leqslant 4$. Below is a sketch of the region $\mathscr{R}$, together with a typical horizontal segment.



$$x_L = 2\sqrt{y}$$

$$x_R = y + 2$$

$$\ell = x_R - x_L = y - 2\sqrt{y} + 2$$

a. The solid obtained by revolving $\mathscr{R}$ about the x-axis can be decomposed into concentric cylinders of radius y and height $y - 2\sqrt{y} + 2$, for $0 \leqslant y \leqslant 4$, so its volume is equal to

$$2\pi \int_0^4 y(y - 2\sqrt{y} + 2)\,dy.$$

b. The solid obtained by revolving $\mathscr{R}$ about about the line defined by $x = -1$ can be decomposed into annuli whose inner radius is $2\sqrt{y} + 1$ and whose outer radius is $y + 3$, for $0 \leqslant y \leqslant 4$, so its volume is equal to

$$\pi \int_0^4 \left\{ (y+3)^2 - (2\sqrt{y}+1)^2 \right\} dy.$$

6. If $y = \frac{1}{12}x^3 + x^{-1}$, then $dy/dx = \frac{1}{4}x^2 - x^{-2}$, and so

$$\begin{aligned}1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \tfrac{1}{16}x^4 - \tfrac{1}{2} + x^{-4} \\ &= \tfrac{1}{16}x^4 + \tfrac{1}{2} + x^{-4} \\ &= \left(\tfrac{1}{4}x^2 + x^{-2}\right)^2.\end{aligned}$$

Therefore, the length of the given curve is equal to

$$\int_1^2 \left(\tfrac{1}{4}x^2 + x^{-2}\right)dx = \left(\tfrac{1}{12}x^3 - x^{-1}\right)\Big|_1^2 = \left(\tfrac{2}{3} - \tfrac{1}{2}\right) - \left(\tfrac{1}{12} - 1\right) = \tfrac{13}{12}.$$

7. Separating variables (and doubling) gives

$$2y\frac{dy}{dx} = \frac{2x}{x^2+1} \qquad \text{and so} \qquad y^2 = \log(x^2+1) + C$$

for some real number C, where $y < 0$ since the initial condition specifies that $y = -4$ if $x = 0$. This initial condition implies that $C = 16$, and so

$$y = -\sqrt{16 + \log(x^2+1)}.$$

8. a. The Maclaurin series of the function f, defined by $f(x) = \sin\left(\frac{1}{2}x\right)$, is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+1}} \cdot \frac{x^{2k+1}}{(2k+1)!} = \tfrac{1}{2}x - \tfrac{1}{8} \cdot \tfrac{1}{6}x^3 + \tfrac{1}{32} \cdot \tfrac{1}{120}x^5 - \cdots$$

and so, since $a_n = f^{(n)}(0)$, $a_1 = \frac{1}{2}$, $a_3 = -\frac{1}{8}$, $a_5 = \frac{1}{32}$, and $a_2 = a_4 = 0$.

b. Since $0 \leqslant |a_n| \leqslant 2^{-n}$, the Squeeze Theorem implies that $\lim a_n = 0$.

c. Since $f(0) = 0$, it follows that

$$\sum_{n=1}^{\infty} \frac{(5\pi)^n a_n}{2^n n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\tfrac{5}{2}\pi\right)^n = \sin\left(\tfrac{5}{4}\pi\right) = -\tfrac{1}{2}\sqrt{2}.$$

9. a. The sum of a series is the limit of its sequence of partial sums, so

$$\sum_{n=1}^{\infty} a_n = \lim s_n = \lim \frac{5n}{2n+1} = \lim \frac{5}{2+1/n} = \tfrac{5}{2}.$$

b. Since $s_n = a_1 + a_2 + \cdots + a_n$, it follows that $a_n = s_n - s_{n-1}$ for $n \geqslant 2$, and so

$$a_3 = s_3 - s_2 = \frac{5 \cdot 3}{2 \cdot 3 + 1} - \frac{5 \cdot 2}{2 \cdot 2 + 1} = \tfrac{15}{7} - 2 = \tfrac{1}{7}.$$

10. a. Let $a_n = \left(1 + 1/n\right)^{3n}$. Since $a_n > 1$ for $n \geqslant 1$, it follows that $\lim a_n \neq 0$, and therefore the series $\sum a_n$ diverges by the vanishing criterion. (It could also be noted that $\lim a_n = e^3 \neq 0$, but this is not necessary.)

b. Since

$$0 < a_n = \frac{5^n + 7^n}{2^n + 9^n} < \frac{2 \cdot 7^n}{9^n} = 2\left(\tfrac{7}{9}\right)^n,$$

if $n \geqslant 1$, and since

$$\sum_{n=1}^{\infty} \left(\tfrac{7}{9}\right)^n = \tfrac{7}{2},$$

the Comparison Test implies that the series $\sum a_n$ is convergent. (It is sufficient to point out that the ratio of the geometric series is positive and smaller than one.)

c. Since

$$0 < a_n = \frac{\cos^2 n}{n\sqrt{n+1}} < \frac{1}{n\sqrt{n}} = n^{-3/2},$$

if $n \geqslant 1$, and since $\sum n^{-3/2}$ is a convergent p-series ($p = \frac{3}{2} > 1$), it follows that $\sum a_n$ is convergent by the Comparison Test.

11. a. If

$$a_n = \frac{3^n (n!)^2}{(2n)!}$$

then, since $a_n > 0$ for $n \geqslant 1$, and

$$\lim \frac{a_{n+1}}{a_n} = \lim \left\{ \frac{3^{n+1}\left((n+1)!\right)^2}{(2n+2)!} \cdot \frac{(2n)!}{3^n (n!)^2} \right\} = \lim \frac{3(n+1)}{2(2n+1)} = \tfrac{3}{4} < 1,$$

the Ratio Test implies that the series $\sum(-1)^n a_n$ is absolutely convergent.

b. Let $a_n = \sin(1/n)$. Since $\vartheta \cos\vartheta < \sin\vartheta < \vartheta$ if $0 < \vartheta < \frac{1}{2}\pi$, it follows that $0 < 1/(2n) < \sin(1/n)$ if $n \geqslant 1$. Therefore, the series $\sum a_n$ diverges with the harmonic series $\sum n^{-1}$ by the Comparison Test. On the other hand, $0 < 1/(n+1) < 1/n < \frac{1}{2}\pi$, and so $0 < a_{n+1} < a_n$, if $n \geqslant 1$, since the sine function is positive and increasing on $(\,0, \frac{1}{2}\pi\,)$. Also, $\lim a_n = \sin 0 = 0$, so the Alternating Series Test implies that the series $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum(-1)^n a_n$ is conditionally convergent. (Instead of noting that $1/(2n) < \sin(1/n)$, one could use the Limit Comparison Test to conclude that $\sum a_n$ diverges with $\sum n^{-1}$.)

12. If

$$\alpha_n = \frac{(2x-3)^n}{3^n\sqrt{2n+3}},$$

then

$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim\left\{\left|\frac{(2x-3)^{n+1}}{3^{n+1}\sqrt{2n+5}} \cdot \frac{3^n\sqrt{2n+3}}{(2x-3)^n}\right|\right\} = \tfrac{1}{3}|2x-3| \lim\sqrt{\frac{2+3/n}{2+5/n}} = \tfrac{1}{3}|2x-3|,$$

so the Ratio Test implies that $\sum \alpha_n$ is absolutely convergent if $|2x-3| < 3$, *i.e.*, $0 < x < 3$, and divergent if $x < 0$ or $x > 3$. It follows that the radius of convergence of $\sum \alpha_n$ is $\frac{3}{2}$.

Next, observe that $\alpha_n = (-1)^n a_n$ if $x = 0$, and $\alpha_n = a_n$ if $x = 3$, where

$$a_n = \frac{1}{\sqrt{2n+3}} > \frac{1}{\sqrt{4n}} = \tfrac{1}{2}n^{-1/2} > 0,$$

if $n \geqslant 2$. The Comparison Test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$ ($p = \frac{1}{2} < 1$), so 3 does not belong to the interval of convergence of $\sum \alpha_n$. On the other hand, $\lim a_n = 0$ and

$$a_n = \frac{1}{\sqrt{2n+3}} > \frac{1}{\sqrt{2n+5}} = a_{n+1} > 0,$$

so the Alternating Series Test implies that the series $\sum(-1)^n a_n$ is convergent; *i.e.*, 0 does belong to the interval of convergence of $\sum \alpha_n$. Therefore, the interval of convergence of $\sum \alpha_n$ is $[\,0, 3\,)$.

13. The binomial expansion gives

$$\begin{aligned} f(x) = \frac{1}{\sqrt{2x+1}} &= (1+2x)^{-1/2} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-k+1)}{k!}(2x)^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\, 1 \cdot 3 \cdots (2k-1)}{k!} x^k \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^k (k!)^2} x^k, \end{aligned}$$

at least if $|2x| < 1$, *i.e.*, $-\frac{1}{2} < x < \frac{1}{2}$. If $x = -\frac{1}{2}$, then this series is $\sum a_k$, and if $x = \frac{1}{2}$ then this seires is $\sum(-1)^k a_k$, where

$$\frac{a_{k+1}}{a_k} = \frac{2k+1}{2(k+1)} = \frac{k+\frac{1}{2}}{k+1}.$$

Since $-1 \leqslant \frac{1}{2} - 1 < 0$, the Gauß Test implies that $\sum(-1)^k a_k$ converges and $\sum a_k$ diverges. Therefore, the interval of convergence of the Maclaurin series of f is $(\,-\frac{1}{2}, \frac{1}{2}\,]$

14. a. If $a_n = (-1)^n n^{-1/2}$ then $\sum a_n$ is convergent by the Alternating Series Test, but $\sum(a_n)^2$ is the harmonic series, which is divergent.

b. Since $\sum a_n$ is convergent, the Vanishing Condition implies that $\lim a_n = 0$, so there is a positive integer N such that $|a_n| < 1$ if $n \geqslant N$. If n is larger than N and k, then $0 \leqslant a_n < 1$, and therefore, $0 < (a_n)^2 < a_n$, so the Comparison Test implies that $\sum(a_n)^2$ is convergent.