1. a. Find the exact value of $\cos\left(\tan^{-1}\left(\dfrac{3}{2x}\right)\right)$.

b. Use your answer to Part a to evaluate

$$\int \cos\left(\tan^{-1}\left(\frac{3}{2x}\right)\right)dx.$$

2. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{11x^2-14x+8}{(2x-1)(x^2+1)}\,dx$ b. $\displaystyle\int_0^{\frac{1}{4}} \frac{\arccos(2x)}{\sqrt{1-4x^2}}\,dx$

c. $\displaystyle\int x^5\cos(x^2)\,dx$ d. $\displaystyle\int \frac{\sin^3(5x)}{\cos(5x)}\,dx$ e. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^4}\,dx$

3. Evaluate each of the following improper integrals.

a. $\displaystyle\int_5^{\infty} \frac{dx}{x^2-10x+29}$ b. $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$

4. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to\frac{1}{6}\pi}\left\{\sec(3x)\sin\left(x-\tfrac{1}{6}\pi\right)\right\}$ b. $\displaystyle\lim_{x\to e^-}(\ln x)^{\frac{2}{1-\ln x}}$ c. $\displaystyle\lim_{x\to 0}\frac{\arctan(2x)}{\arctan(5x)}$

5. Find the area of the region enclosed by the graphs of

$$y=\sqrt{x+1} \qquad \text{and} \qquad y=\tfrac{1}{2}(x+1).$$

6. Let $\mathscr{R}$ be the region bounded by the graphs of $y=-x^2+3x$ and $y=x^2$. Set up, but do not evaluate, an integral which represents the volume of the solid obtained by rotating the region $\mathscr{R}$ about

a. the y-axis, b. the line defined by $y=-1$.

7. Find the length of the curve defined by

$$y=2\ln\left(\cos\tfrac{1}{2}x\right), \quad \tfrac{1}{3}\pi \leqslant x \leqslant \tfrac{1}{2}\pi.$$

8. Solve the differential equation

$$x^2y'+2xy=3x,$$

given $y(1)=1$ and $x>0$.

9. Find the limit of the sequence whose general term is

$$a_n=\frac{4}{9^n}+3\arctan\left(\ln(n^2)\right),$$

or else explain why the sequence diverges.

10. Determine whether each series is convergent. Justify all assertions carefully.

a. $\displaystyle\sum_{n=0}^{\infty}\frac{e^n}{1+e^{2n}}$ b. $\displaystyle\sum_{k=1}^{\infty}\left(1-\frac{1}{\sqrt[3]{k^2}}\right)^k$

c. $\displaystyle\sum_{j=0}^{\infty}\left(\frac{2j+1}{3j+1}\right)^j$ d. $\displaystyle\sum_{m=0}^{\infty}\left(1+\frac{7}{4^m}\right)$

11. For each of the series below, determine whether it is absolutely convergent, conditionally convergent or divergent. Justify all assertions carefully.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{\ln(4n)}$ b. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k5^kk^2}{(2k)!}$

12. Find the sum of each series, or explain why the series diverges.

a. $\displaystyle\sum_{i=1}^{\infty}\left\{\arccos\left(\frac{1}{i+1}\right)-\arccos\left(\frac{1}{i+2}\right)\right\}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n-1}}{5^n}$ c. $\displaystyle\sum_{k=1}^{\infty}\frac{1+k}{2^kk}$

13. Suppose that the power series

$$\sum_{n=0}^{\infty}c_nx^n$$

converges if $x=-5$ and diverges if $x=7$. Investigate the convergence of each of the following series. Justify your answers.

a. $\displaystyle\sum_{n=0}^{\infty}(-1)^nc_n4^n$ b. $\displaystyle\sum_{n=0}^{\infty}c_n5^n$ c. $\displaystyle\sum_{n=0}^{\infty}c_n$ d. $\displaystyle\sum_{n=0}^{\infty}(-1)^nc_n8^n$

14. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n3^{2n}}{4n+1}(x-2)^n.$$

15. Find the Taylor series centred at 5 of the function f defined by

$$f(x)=\frac{1}{2-x}.$$

Write the first five non-zero terms of the series explicitly, and give the interval of convergence of the series.

1. a. Let $\vartheta = \tan^{-1}\left(\dfrac{3}{2x}\right)$; so $\tan\vartheta = \dfrac{3}{2x}$ and $-\frac{1}{2}\pi < \vartheta < \frac{1}{2}\pi$. Since

$$\cos^2\vartheta = \frac{1}{\sec^2\vartheta} = \frac{1}{1+\tan^2\vartheta} = \frac{1}{1+\left(\dfrac{3}{2x}\right)^2} = \frac{(2x)^2}{4x^2+9},$$

and since $\cos\vartheta > 0$ if $-\frac{1}{2}\pi < \vartheta < \frac{1}{2}\pi$, it follows that

$$\cos\left(\tan^{-1}\left(\frac{3}{2x}\right)\right) = \frac{2|x|}{\sqrt{4x^2+9}},$$

provided $x \neq 0$. (The function has a removable discontinuity at the origin.)

b. Since

$$\frac{d}{dx}\left\{\sqrt{4x^2+9}\right\} = \frac{4x}{\sqrt{4x^2+9}},$$

it follows that

$$\int \cos\left(\tan^{-1}\left(\frac{3}{2x}\right)\right)dx = \frac{|x|}{2x}\sqrt{4x^2+9}+C,$$

on any interval which does not contain zero (and approaches $C \pm \frac{3}{2}$ as $x \to 0^{\pm}$).

2. a. The resolution into partial fractions of the integrand has the form

$$\frac{11x^2-14x+8}{(2x-1)(x^2+1)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+1},$$

and $A = 3$ is obtained by covering. Clearing denominators gives

$$11x^2-14x+8 = A(x^2+1)+(Bx+C)(2x-1),$$

and comparing the coefficients of x^2 yields $A+2B = 11$, so $B = 4$. Comparing the constant terms yields gives $A - C = 8$, so $C = -5$. Therefore,

$$\int \frac{11x^2-14x+8}{(2x-1)(x^2+1)}\,dx = \int\left\{\frac{3}{2x-1}+\frac{4x-5}{x^2+1}\right\}dx$$
$$= \tfrac{3}{2}\log|2x-1| + 2\log(x^2+1) - 5\arctan x + C.$$

b. If $t = \arccos(2x)$, then

$$\int_0^{\frac{1}{4}} \frac{\arccos(2x)}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} t\,dt = \tfrac{1}{4}t^2\,\Big|_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} = \tfrac{1}{4}\pi^2\left(\tfrac{1}{4}-\tfrac{1}{9}\right) = \tfrac{5}{144}\pi^2.$$

c. If $t = x^2$, then repeated partial integration gives

$$\int x^5\cos(x^2)\,dx = \frac{1}{2}\int t^2\cos t\,dt$$
$$= \tfrac{1}{2}\big(t^2\sin t - 2t(-\cos t) + 2(-\sin t)\big) + C$$
$$= \tfrac{1}{2}(x^4-2)\sin(x^2) + x^2\cos(x^2) + C.$$

d. If $t = \cos(5x)$ then

$$\int \frac{\sin^3(5x)}{\cos(5x)}\,dx = \frac{1}{5}\int\frac{t^2-1}{t}\,dt = \tfrac{1}{10}t^2 - \tfrac{1}{5}\log|t| + C$$
$$= \tfrac{1}{10}\cos^2(5x) - \tfrac{1}{5}\log|\cos(5x)| + C.$$

e. If $0 < x \leqslant 1$ and $t = x^{-1}\sqrt{1-x^2} = \sqrt{x^{-2}-1}$, then $-t\,dt = x^{-3}\,dx$, so

$$\int\frac{\sqrt{1-x^2}}{x^4}\,dx = \int\sqrt{x^{-2}-1}\cdot\frac{dx}{x^3} = -\int t^2\,dt = -\tfrac{1}{3}t^3 + C$$
$$= -\frac{(1-x^2)^{3/2}}{3x^3} + C.$$

If $-1 \leqslant x < 0$ the same result follows by symmetry (since the derivative of an odd function is even).

3. a. Since $x^2-10x+29 = (x-5)^2+4$, integrating by inspection gives

$$\int_5^\infty \frac{dx}{x^2-10x+29} = \lim_{\alpha\to\infty}\tfrac{1}{2}\arctan\left(\tfrac{1}{2}(x-5)\right)\Big|_5^\alpha = \tfrac{1}{4}\pi.$$

b. Since the antiderivative $2\sqrt{\tan x}$ of the integrand is continuous on $\left[0, \frac{1}{4}\pi\right]$, it follows that

$$\int_0^{\frac{1}{4}\pi}\frac{\sec^2 x}{\sqrt{\tan x}}\,dx = 2\sqrt{\tan\tfrac{1}{4}\pi} - 2\sqrt{\tan 0} = 2.$$

4. a. Revising the expression in the limit and then applying l'Hôpital's Rule once gives

$$\lim_{x\to\frac{1}{6}\pi}\frac{\sin\left(x-\frac{1}{6}\pi\right)}{\cos(3x)} = \lim_{x\to\frac{1}{6}\pi}\frac{\cos\left(x-\frac{1}{6}\pi\right)}{-3\sin(3x)} = -\tfrac{1}{3}.$$

b. The expression in the limit is equal to e^y, where

$$y = \frac{2\log\log x}{1-\log x} = \frac{-2\log\log x}{x-e}\cdot\frac{x-e}{\log x - 1} \to -\frac{2}{e}\cdot e = -2$$

as $x \to e$, by the definition of the derivative of a function. Therefore, the limit in question is equal to e^{-2}.

c. Since

$$\lim_{x\to 0}\frac{\arctan(\alpha x)}{x} = \alpha,$$

by the definition of the derivative of a function, the limit in question is equal to $\frac{2}{5}$.

5. The curves meet where $x+1 = 2\sqrt{x+1}$, or $(\sqrt{x+1}-2)\sqrt{x+1} = 0$, *i.e.*, where x is -1 or 3. If $-1 < x < 3$ then $(\sqrt{x+1}-2)\sqrt{x+1} < 0$, so the area of the region enclosed by the curves is equal to

$$\int_{-1}^3\left\{\sqrt{x+1}-\tfrac{1}{2}(x+1)\right\}dx = \left\{\tfrac{2}{3}(x+1)^{3/2} - \tfrac{1}{4}(x+1)^2\right\}\Big|_{-1}^3 = \tfrac{4}{3}.$$

6. The curves meet where $x^2 = -x^2+3x$, or $x(2x-3) = 0$, *i.e.*, where $x = 0$ or $x = \frac{3}{2}$. If $0 < x < 3$, $x(2x-3)$ is negative, so the graph of $y = x^2$ is below the graph of $y = -x^2+3x$; hence, $\mathscr{R} = \{\,(x,y) \colon 0 \leqslant x \leqslant \frac{3}{2} \text{ and } 0 \leqslant 3x - 2x^2\,\}$.

a. The solid obtained by revolving $\mathscr{R}$ about the y-axis can be decomposed into concentric cylindrical shells of radius x and height $3x-2x^2$, for $0 \leqslant x \leqslant \frac{3}{2}$, so its volume is equal to

$$2\pi\int_0^{\frac{3}{2}} x(3x-2x^2)\,dx.$$

b. The solid obtained by revolving $\mathscr{R}$ about the line defined by $y = -1$ can be decomposed into annuli of inner radius $1+x^2$ and outer radius $1+3x-x^2$, for $0 \leqslant x \leqslant \frac{3}{2}$, so its volume is equal to

$$\pi\int_0^{\frac{3}{2}}\left\{(1+3x-x^2)^2 - (1+x^2)^2\right\}dx.$$

7. If $y = 2\ln(\cos\frac{1}{2}x)$, then

$$\frac{dy}{dx} = -\tan\tfrac{1}{2}x, \qquad \text{and so} \qquad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sec\tfrac{1}{2}x,$$

at least if $0 < x < \pi$ (among other possibilities). So the length of the curve in question is equal to

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\sec\tfrac{1}{2}x\,dx = 2\log\left(\sec\tfrac{1}{2}x + \tan\tfrac{1}{2}x\right)\Big|_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}$$
$$= 2\log\frac{\sqrt{2}+1}{\frac{2}{3}\sqrt{3}+\frac{1}{3}\sqrt{3}} = 2\log\frac{1+\sqrt{2}}{\sqrt{3}}$$
$$= \log\left(1+\tfrac{2}{3}\sqrt{2}\right).$$

8. The equation $x^2y' + 2xy = 3x$ is equivalent to

$$\frac{d}{dx}(x^2y) = 3x, \qquad \text{or} \qquad x^2y = \tfrac{3}{2}x^2 + C,$$

on any interval where x does not vanish. The requirement that $y = 1$ if $x = 1$ implies that $C = -\frac{1}{2}$, and so the solution of the equation is

$$x^2y = \tfrac{3}{2}x^2 - \tfrac{1}{2}, \qquad \text{or} \qquad y = \tfrac{3}{2} - \tfrac{1}{2}x^{-2},$$

where $x > 0$.

9. As $n \to \infty$, $4/9^n \to 0$ and $\ln(n^2) \to \infty$, so $\arctan\left(\ln(n^2)\right) \to \frac{1}{2}\pi$. Therefore, the sequence in question converges to $\frac{3}{2}\pi$.

10. a. Since

$$0 < \frac{e^n}{1+e^{2n}} < \frac{e^n}{e^{2n}} = e^{-n},$$

if $n \geqslant 0$, and since $\sum e^{-n}$ is a convergent geometric series ($|r| = e^{-1} < 1$), the Comparison Test implies that the series

$$\sum_{n=0}^\infty\frac{e^n}{1+e^{2n}}$$

is convergent.

b. Let $a_k = \big(1 - k^{-2/3}\big)^k$; then

$$0 < a_k = e^{k\log(1-k^{-2/3})} = e^{-k^{1/3} - k^{-1/3}/2 - k^{-1}/3 - \cdots} < e^{-k^{1/3}}$$

if $k \geqslant 2$, using the Maclaurin expansion

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^k}{k}$$

(which is valid if $-1 \leqslant x < 1$) with $x = k^{-2/3}$. If $t = k^{1/3}$, then

$$\lim \frac{e^{-k^{1/3}}}{k^{-4/3}} = \lim_{t\to\infty} \frac{t^4}{e^t} = \lim_{t\to\infty} \frac{24}{e^t} = 0,$$

by four applications of l'Hôpital's Rule, so the Limit Comparison Test implies that the series $\sum e^{-k^{1/3}}$ converges with the p-series $\sum k^{-4/3}$ ($p = \frac{4}{3} > 1$). Hence, by the first displayed inequality, the Comparison Test implies that $\sum a_k$ converges with $\sum e^{-k^{1/3}}$.

c. If

$$a_j = \left(\frac{2j+1}{3j+1}\right)^j, \qquad \text{then} \qquad \lim (a_j)^{1/j} = \lim \frac{2 + 1/j}{3 + 1/j} = \tfrac{2}{3},$$

which is smaller than 1, so the Root Test implies that $\sum a_j$ is a convergent series.

d. Since $\lim\big(1 + 7/4^m\big) = 1$, the vanishing criterion implies that the series $\sum\big(1 + 7/4^m\big)$ is divergent.

11. a. If

$$a_n = \frac{1}{\log(4n)},$$

then, since $0 < \log(4n) \leqslant 2\log n < 2n/e < n$, if $n \geqslant 4$ (from Calculus I the largest value of $(\log x)/x$ is $(\log e)/e = 1/e$), and since the harmonic series $\sum n^{-1}$ is divergent, the Comparison Test implies that $\sum a_n$ is divergent. However, $\{\log(4n)\}_{n\geqslant 1}$ is positive, increasing and unbounded, so the Alternating Series Test implies that $\sum(-1)^n a_n$ is convergent. Hence, the series $\sum(-1)^n a_n$, *i.e.*, $\sum\big\{\cos(\pi n) a_n\big\}$, is conditionally convergent.

b. If

$$a_k = \frac{(-1)^k 5^k k^2}{(2k)!},$$

then

$$\lim\left|\frac{a_{k+1}}{a_k}\right| = \lim\left\{\frac{5^{k+1}(k+1)^2}{(2k+2)!} \cdot \frac{(2k)!}{5^k k^2}\right\} = \tfrac{5}{2} \lim \frac{(1 + 1/k)^2}{(k+1)(2k+1)} = 0,$$

so the Ratio Test implies that $\sum a_k$ is absolutely convergent.

12. a. The series in question is apparently telescoping, and the general term of its sequence of partial sums is equal to

$$\arccos\big(\tfrac{1}{2}\big) - \arccos\left(\frac{1}{n+3}\right),$$

which converges to $\arccos\big(\frac{1}{2}\big) - \arccos(0) = \frac{1}{3}\pi - \frac{1}{2}\pi = -\frac{1}{6}\pi$.

b. The series in question is a geometric series with first term $\frac{1}{5}$ and ratio $\frac{2}{5}$, so it is convergent and its sum is equal to $\frac{1}{5}/\big(1 - \frac{2}{5}\big) = \frac{1}{3}$.

c. Separating the terms of the series gives

$$\sum_{k=1}^{\infty} \frac{1+k}{2^k k} = \sum_{k=1}^{\infty} \big(\tfrac{1}{2}\big)^k + \sum_{k=1}^{\infty} \frac{k}{2^k k},$$

provided the terms on the right both converge. The first term on the right is a convergent geometric series whose sum is $\frac{1}{2}/\big(1 - \frac{1}{2}\big) = 1$. The second term is obtained from the Maclaurin expansion of $-\log(1-x)$ by letting $x = \frac{1}{2}$, so its sum is $-\log\big(1 - \frac{1}{2}\big) = \log 2$ (the Maclaurin series is displayed in the solution to Question 10b). Therefore, the sum of the series in question is $\log(2e)$.

13. Let $f(x)$ denote the power series in question, whose interval of convergence includes the interval $[-5, 5)$, is included in the interval $[-7, 7)$, and may be equal to either of these intervals.

a. The series in question is $f(-4)$, which converges since -4 belongs to $[-5, 5)$.

b. The series in question is $f(5)$, which may converge or diverge, since 5 belongs to $[-7, 7)$, but does not belong to $[-5, 5)$.

c. The series in question is $f(1)$, which converges since 1 belongs to $[-5, 5)$.

d. The series in question is $f(-8)$, which diverges because -8 does not belong to $[-7, 7)$.

14. If u_n denotes the general term of the given power series, then (provided $x \neq 2$)

$$\lim\left|\frac{u_{n+1}}{u_n}\right| = 9|x-2| \lim \frac{n + 1/4}{n + 3/4} = 9|x-2|,$$

so the Ratio Test implies that $\sum u_n$ converges if $9|x-2| < 1$, *i.e.*, $\frac{17}{9} < x < \frac{19}{9}$, and diverges if $x < \frac{17}{9}$ or $x > \frac{19}{9}$. If $x = \frac{17}{2}$ then u_n is equal to

$$a_n = \frac{1}{4n+1} \geqslant \tfrac{1}{5} n^{-1},$$

provided $n \geqslant 1$, and $\sum n^{-1}$ is a divergent p-series, so $\sum u_n$ diverges if $x = \frac{17}{9}$. If $x = \frac{19}{9}$ then $\sum u_n$ is $\sum(-1)^n a_n$, which converges by the Alternating Series Test, since a_n is positive if $n \geqslant 1$, decreasing (since it is the reciprocal of an increasing function) and $\lim a_n = 0$. So the interval of convergence of $\sum u_n$ $\big(\frac{17}{9}, \frac{19}{9}\big]$, and the radius of convergence of $\sum u_n$ is $\frac{1}{9}$.

15. Revising the expression defining f and then expanding the resulting geometric series gives

$$\begin{aligned} f(x) = \frac{1}{2-x} &= -\frac{1}{3} \cdot \frac{1}{1 - \big(-(x-5)/3\big)} = -\tfrac{1}{3} \sum_{k=0}^{\infty} \big(-(x-5)/3\big)^k \\ &= \sum_{k=0}^{\infty} \big(-\tfrac{1}{3}\big)^{k+1} (x-5)^k \\ &= -\tfrac{1}{3} + \tfrac{1}{9}(x-5) - \tfrac{1}{27}(x-5)^2 + \tfrac{1}{81}(x-5)^3 - \tfrac{1}{243}(x-5)^4 + \cdots \end{aligned}$$

provided $\big|\frac{1}{3}(x-5)\big| < 1$, *i.e.*, $2 < x < 8$ (which gives the interval of convergence of the series).