1. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{x+3}{\sqrt{x^2+6x+5}}\,dx$ b. $\displaystyle\int_0^{\frac{1}{2}\pi} \sin(2x)\sin(3x)\,dx$

c. $\displaystyle\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx$ d. $\displaystyle\int \frac{e^{8/x}}{x^2(2+e^{8/x})}\,dx$

e. $\displaystyle\int \frac{30x+14}{(3x-2)^2(x^2+2x+2)}\,dx$ f. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^3}\,dx$

g. $\displaystyle\int \frac{2}{(1-x)^2}\log\left(\frac{1+x}{1-x}\right)dx$

2. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 1^+}\left\{\log(x^5-1)-\log(\sqrt{x}-1)\right\}$

b. $\displaystyle\lim_{x\to 0}\big(\arcsin(x)\cot(x)\big)$ c. $\displaystyle\lim_{x\to\infty}\left(1-\frac{3}{x^2}\right)^{x^2}$

3. Evaluate each of the following improper integrals.

a. $\displaystyle\int_1^\infty \frac{3}{(2x-1)(x+1)}\,dx$ b. $\displaystyle\int_0^4 \frac{1}{(x-3)^2}\,dx$

4. Find the length of the curve defined by

$$y=\tfrac{4}{5}x^{5/4},\qquad 0\leqslant x\leqslant 9.$$

5. Let $\mathscr{R}$ be the region bounded by the graph of

$$y=\frac{1}{\sqrt{x^2+4}}$$

and the x-axis, between $x=0$ and $x=2$.

a. Find the volume of the solid obtained by rotating $\mathscr{R}$ about the y-axis.

b. Set up, *but do not evaluate*, an integral which represents the volume of the solid obtained by rotating $\mathscr{R}$ about: i. the x-axis; ii. the line $y=2$.

6. Solve the differential equation

$$(4+x^2)^2\frac{dy}{dx}=-2\pi x(1+y^2),$$

subject to the initial condition $y(0)=\frac{1}{3}\sqrt{3}$.

7. Determine whether or not each sequence $\{a_n\}$ is convergent. If a sequence is convergent, find its limit. Justify your answers.

a. $a_n=\dfrac{3}{2^n}+\cos(e^{-n})$ b. $a_n=\dfrac{n!}{3^n}$

8. It is given that

$$a_1+a_2+\cdots+a_n=2+\frac{(-1)^n}{2^n},\qquad \text{for}\qquad n\geqslant 1.$$

a. Evaluate $\displaystyle\sum_{n=1}^\infty a_n$. b. Find a_n for $n\geqslant 2$.

9. Determine whether each series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^\infty ne^{-n^2}$ b. $\displaystyle\sum_{n=1}^\infty \frac{n}{2n^2-\cos n}$ c. $\displaystyle\sum_{n=1}^\infty \frac{2^{2n-1}-1}{n+3^n}$

10. For each series below, determine whether it is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=0}^\infty \frac{3n(-3)^n}{(n+3)!}$ b. $\displaystyle\sum_{n=1}^\infty (\arctan n-2)^{n/3}$ c. $\displaystyle\sum_{n=1}^\infty (-1)^n\log\left(1+\frac{1}{n}\right)$

11. Find the Taylor series of $x\log x$ centred at 5. Write the first five terms of the series explicitly and express the series in appropriate sigma notation. What is the interval of convergence of the series?

12. Find the interval of convergence of the power series

$$\sum_{n=1}^\infty \frac{(-1)^n(3x-2)^n}{2^{3n}(3n+2)^2}.$$

1. a. By inspection,
$$\int \frac{x+3}{\sqrt{x^2+6x+5}}\,dx = \sqrt{x^2+6x+5}+C.$$
b. Since $2\sin(2x)\sin(3x) = \cos(x)-\cos(5x)$, it follows that
$$\int_0^{\frac12\pi} \sin(2x)\sin(3x)\,dx = \tfrac12\big(\sin(x)-\tfrac15\sin(5x)\big)\Big|_0^{\frac12\pi} = \tfrac25.$$
Alternatively, let $\mathscr{I}$ denote the integral in question and integrate by parts twice:
$$\mathscr{I} = -\tfrac12\cos(2x)\sin(3x)\Big|_0^{\frac12\pi} + \tfrac32\int_0^{\frac12\pi}\cos(2x)\cos(3x)\,dx$$
$$= -\tfrac12 + \tfrac34\sin(2x)\cos(3x)\Big|_0^{\frac12\pi} + \tfrac94\int_0^{\frac12\pi}\sin(2x)\sin(3x)\,dx.$$
This implies that $-\frac54\mathscr{I} = -\frac12$, or $\mathscr{I} = \frac25$.
c. If $t = \cos(\sqrt{x})$, then
$$-2\,\frac{dt}{dx} = \frac{\sin(\sqrt{x})}{\sqrt{x}}, \qquad \text{and} \qquad \sin^2(\sqrt{x}) = 1-t^2,$$
so the integral in question is equal to
$$2\int(t^2-1)\,dt = \tfrac23t^3-2t+C = \tfrac23\cos^3(\sqrt{x})-2\cos(\sqrt{x})+C.$$
d. If $t = 2+e^{8/x}$, then $-\frac18\,dt/dx = e^{8/x}/x^2$, so the the integral becomes
$$-\tfrac18\int\frac{dt}{t} = -\tfrac18\log t + C = -\tfrac18\log\big(2+e^{8/x}\big)+C.$$
e. The resolution into partial fractions of the integrand has the form
$$\frac{A}{3x-2}+\frac{B}{(3x-2)^2}+\frac{Cx+D}{x^2+2x+2} = \frac{30x+14}{(3x-2)^2(x^2+2x+2)}.$$
Multiplying by $(3x-2)^2$ and evaluating at $\frac23$ gives $B=9$. Next, after clearing denominators the right side is $30x+14$ and the left side is (expanding the square)
$$A(3x-2)(x^2+2x+2)+B(x^2+2x+2)+(Cx+D)(9x^2-12x+4).$$
Comparing coefficients of x^3 gives $3A+9C=0$, or $A=-3C$. Comparing coefficients of x^2 gives $4A+B-12C+9D=0$. Since $B=9$ and $A=-3C$, this gives $-24C+9D=-9$, or
$$8C-3D=3. \tag{1}$$
Comparing coefficients of x gives $2A+2B+4C-12D=30$. Since $B=9$ and $A=-3C$, this gives $-2C-12D=12$, or
$$C+6D=-6. \tag{2}$$
Adding twice (1) to (2) yields $17C=0$, which implies that $C=0$ and $D=-1$. Since $A=-3C$, it follows that $A=0$. Since $x^2+2x+2=(x+1)^2+1$, the integral in question is equal to
$$\int\left\{\frac{9}{(3x-2)^2}-\frac{1}{x^2+2x+2}\right\} = -\frac{3}{3x-2}-\arctan(x+1)+C.$$
f. Integrating by parts, and then by inspection, gives
$$\int\frac{\sqrt{x^2-1}}{x^3}\,dx = -\frac{\sqrt{x^2-1}}{2x^2}+\frac12\int\frac{dx}{x\sqrt{x^2-1}}$$
$$= -\frac{\sqrt{x^2-1}}{2x^2}+\tfrac12\operatorname{arcsec}x+C.$$
g. If $z=(1+x)/(1-x)$, then $\frac{dz}{dx}=2/(1-x)^2$, so partial integration gives
$$\int\log z\,dz = z\log z - z + C = \frac{1+x}{1-x}\log\left(\frac{1+x}{1-x}\right)-\frac{1+x}{1-x}+C.$$

2. a. The definition of the derivative, with $t=\sqrt{x}$, gives
$$\lim_{x\to1^+}\big(\log(x^5-1)-\log(\sqrt{x}-1)\big) = \lim_{t\to1^+}\log\left(\frac{t^{10}-1}{t-1}\right) = \log 10.$$
b. Multiplying and dividing by x, and then factorizing the limit in question, gives
$$\lim_{x\to0}\big(\arcsin(x)\,\cot(x)\big) = \left(\lim_{t\to0}\frac{t}{\sin t}\right)\left(\lim_{x\to0}\frac{x}{\sin x}\right)\left(\lim_{x\to0}\cos x\right) = 1,$$
where $t=\arcsin x$ (since $(\sin\vartheta)/\vartheta\to1$ as $\vartheta\to0$ from Calculus I).
c. Since $\lim_{t\to0}(1+t)^{1/t}=e$, it follows (letting $t=-3/x^2$) that
$$\lim_{x\to\infty}\left(1-\frac{3}{x^2}\right)^{x^2} = \left(\lim_{t\to0^-}(1+t)^{1/t}\right)^{-3} = e^{-3}.$$

3. a. Resolving the integrand into partial fractions and integrating by inspection gives
$$\int_1^\infty\left\{\frac{2}{2x-1}-\frac{1}{x+1}\right\}dx = \lim_{t\to\infty}\log\left(\frac{2x-1}{x+1}\right)\Big|_1^t = 2\log 2.$$
b. Since
$$\int_3^4\frac{dx}{(x-3)^2} = \int_0^1\frac{dx}{x^2}$$
is a divergent integral in a standard scale ($p=2$, for an infinite discontinuity at the origin), it follows that the integral in question diverges (to ∞).

4. If $y=\frac45x^{5/4}$ then $\frac{dy}{dx}=x^{1/4}$, so the length of the curve is
$$s = \int_0^9\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^9\sqrt{1+\sqrt{x}}\,dx.$$
If $t=\sqrt{1+\sqrt{x}}$, then $x=(t^2-1)^2$ and $\frac{dx}{dt}=4t(t^2-1)$, so
$$s = 4\int_1^2 t^2(t^2-1)\,dt = 4\big(\tfrac15t^5-\tfrac13t^3\big)\Big|_1^2 = 4\big(\tfrac{31}{5}-\tfrac73\big) = \tfrac{232}{15}.$$

5. a. The solid obtained by revolving the region $\mathscr{R}$ about the y-axis consists of concentric cylinders of radius x and height $1/\sqrt{x^2+4}$, for $0\leqslant x\leqslant 2$; its volume is equal to
$$2\pi\int_0^2\frac{x}{\sqrt{x^2+4}}\,dx = 2\pi\sqrt{x^2+4}\Big|_0^2 = 4\pi(\sqrt2-1).$$
b. i. The solid obtained by revolving the region $\mathscr{R}$ about the x-axis consists of circular discs of radius $1/\sqrt{x^2+4}$, for $0\leqslant x\leqslant 2$; its volume is equal to
$$\pi\int_0^2\frac{dx}{x^2+4}.$$
ii. The solid obtained by revolving the region $\mathscr{R}$ about the line defined by $y=2$ consists of annuli of inner radius $2-1/\sqrt{x^2+4}$ and outer radius 2, for $0\leqslant x\leqslant 2$; its volume is equal to
$$\pi\int_0^2\left\{4-\left(2-\frac{1}{\sqrt{x^2+4}}\right)^2\right\}dx.$$

6. Separating variables gives
$$\frac{1}{1+y^2}\frac{dy}{dx} = \frac{-2\pi x}{(x^2+4)^2}, \quad \text{and so} \quad \arctan y = \pi(x^2+4)^{-1}+\alpha,$$
for some real number α. If $y(0)=\frac13\sqrt3$, then $\alpha=\frac16\pi-\frac14\pi=-\frac1{12}\pi$; hence
$$y = \tan\big(\pi(x^2+4)^{-1}-\tfrac1{12}\pi\big).$$

7. a. Since $\lim r^n=0$ if $|r|<1$, it follows that $\lim a_n = 0+\cos(0)=1$.
b. As $a_n = \dfrac{1\cdot2\cdot3\cdots n}{3\cdot3\cdot3\cdots3}\geqslant\frac{2}{27}n$, if $n\geqslant4$, the sequence $\{a_n\}$ diverges to ∞.

8. a. Using the given summation formula,
$$\sum_{n=1}^\infty a_n = \lim\{a_1+a_2+\cdots+a_n\} = \lim\{2+\big(-\tfrac12\big)^n\} = 2,$$
since $\lim r^n=0$ if $|r|<1$.
b. If $n\geqslant2$, then
$$a_n = (a_1+a_2+\cdots+a_{n-1}+a_n)-(a_1+a_2+\cdots+a_{n-1})$$
$$= \big(-\tfrac12\big)^n-\big(-\tfrac12\big)^{n-1}$$
$$= 3\big(-\tfrac12\big)^n.$$

9. a. If $n > 2$ then $n < e^n$ and $n - n^2 < -n$, so

$$0 < ne^{-n^2} < e^{n-n^2} < e^{-n}. \qquad \text{But} \quad \sum_{n=1}^{\infty} e^{-n} = \frac{1}{e-1},$$

so the comparison test implies that the series $\sum ne^{-n^2}$ is convergent.

b. If $n \geqslant 1$ then $2n^2 - \cos n < 3n^2$, and therefore

$$a_n = \frac{n}{2n^2 - \cos n} > \frac{n}{3n^2} = \frac{1}{3n} > 0.$$

Hence, the comparison test implies that $\sum a_n$ diverges with the harmonic series.

c. If $n \geqslant 1$, then

$$a_n = \frac{2^{2n-1} - 1}{n + 3^n} > \frac{4^n \cdot \frac{1}{4}}{2 \cdot 3^n} = \tfrac{1}{8}\left(\tfrac{4}{3}\right)^n. \qquad \text{But} \quad \lim\left(\tfrac{4}{3}\right)^n = \infty,$$

so the Vanishing Condition implies that the series $\sum a_n$ is divergent.

10. a. If

$$a_n = \frac{3n(-3)^n}{(n+3)!},$$

then

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left\{\left(1 + \frac{1}{n}\right)\frac{1}{n+4}\right\} = 0 < 1,$$

so the ratio test implies that the series $\sum a_n$ is absolutely convergent.

b. If $a_n = (\arctan n - 2)^{n/3}$, then

$$\lim \sqrt[n]{|a_n|} = \lim(2 - \arctan n)^{1/3} = \left(2 - \tfrac{1}{2}\pi\right)^{1/3} < 1,$$

so the root test implies that the series $\sum a_n$ is absolutely convergent.

c. It is immediate from the definition of the logarithm that if $n \geqslant 1$, then

$$0 < \frac{1}{n+1} < a_n < \frac{1}{n}, \qquad \text{where} \qquad a_n = \log\left(1 + \frac{1}{n}\right).$$

Hence, the comparison test implies that $\sum a_n$ diverges with the harmonic series. The displayed inequality also implies that $0 < a_{n+1} < a_n$ if $n \geqslant 1$, and that $\lim a_n = 0$. So the alternating series test implies that $\sum(-1)^n a_n$ converges. Therefore, $\sum(-1)^n a_n$ is conditionally convergent.

11. Properties of the logarithm, and the Maclaurin expansion of $\log(1+x)$, give

$$\begin{aligned} x \log x &= \left(5 + (x-5)\right)\left(\log 5 + \log(1 + \tfrac{1}{5}(x-5))\right) \\ &= 5\log 5 + (\log 5)(x-1) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{5^k k}(x-5)^k\left(5 + (x-5)\right). \end{aligned}$$

Now

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{5^k k}(x-5)^k\left(5 + (x-5)\right) \\ &= (x-5) + \sum_{k=2}^{\infty}\left\{\frac{(-1)^{k-1}}{5^{k-1}k} + \frac{(-1)^k}{5^{k-1}(k-1)}\right\}(x-5)^k \\ &= (x-5) + \sum_{k=2}^{\infty} \frac{(-1)^k}{5^{k-1}k(k-1)}(x-5)^k, \end{aligned}$$

and $\log 5 + 1 = \log(5e)$, so the required expansion is

$$5\log 5 + \log(5e)(x-5) + \sum_{k=2}^{\infty} \frac{(-1)^k}{5^{k-1}k(k-1)}(x-5)^k,$$

which is valid *at least* if $-1 < \frac{1}{5}(x-5) \leqslant 1$, *i.e.*, $0 < x \leqslant 10$. If $x = 0$, then the series in question is a convergent telescoping series:

$$\begin{aligned} 5\log 5 - 5\log(5e) + \sum_{k=2}^{\infty} \frac{5}{k(k-1)} &= -5 + 5\sum_{k=2}^{\infty}\left\{\frac{1}{k-1} - \frac{1}{k}\right\} \\ &= -5 + 5\lim\left\{1 - \frac{1}{k}\right\} = 0. \end{aligned}$$

So the interval of convergence of the Taylor series of $x\log x$ centred at 5 is $[\,0, 10\,]$. The sum of the first five terms of the series is

$$5\log 5 + \log(5e)(x-5) + \tfrac{1}{10}(x-5)^2 - \tfrac{1}{150}(x-5)^3 + \tfrac{1}{1500}(x-5)^4.$$

12. Let

$$\alpha_n = \frac{(-1)^n(3x-2)^n}{2^{3n}(3n+2)^2}.$$

If $x \neq \frac{2}{3}$, then

$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \tfrac{1}{8}|3x-2| \lim\left(\frac{3n+2}{3n+5}\right)^2 = \tfrac{1}{8}|3x-2|,$$

so the ratio test implies that $\sum \alpha_n$ is absolutely convergent if $|3x-2| < 8$, *i.e.*$-2 < x < \frac{10}{3}$, and divergent if $x < -2$ or $x > \frac{10}{3}$. If $x = -2$ or $x = \frac{10}{3}$, then

$$|\alpha_n| = \frac{1}{(3n+2)^2} < \tfrac{1}{9}n^{-2}$$

if $n \geqslant 1$. Since $\sum n^{-2}$ is a convergent p-series ($p = 2 > 1$), the comparison test implies that the series $\sum \alpha_n$ is convergent. Hence, the interval of convergence of $\sum \alpha_n$ is $\left[\,-2, \frac{10}{3}\,\right]$.