1. Evaluate each of the folliwng integrals.

a. $\displaystyle\int \frac{7x-6}{(x+1)(4x^2+9)}\,dx$ b. $\displaystyle\int e^x \sec^3(e^x)\tan^3(e^x)\,dx$

c. $\displaystyle\int \sqrt{5+4x-x^2}\,dx$ d. $\displaystyle\int_0^{\frac{1}{6}} \arccos(3x)\,dx$

e. $\displaystyle\int \frac{x+\cos\sqrt{x+2}}{\sqrt{x+2}}\,dx$ f. $\displaystyle\int \frac{1}{x^3}\sin(1/x)\,dx$

2. Calculate each of the following limits.

a. $\displaystyle\lim_{x\to 0^+}(e^x+x)^{2/x}$ b. $\displaystyle\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$

c. $\displaystyle\lim_{x\to\infty}\left\{\ln(3x^2+5)-\ln(x^2+1)\right\}$

3. Determine whether or the following improper integrals converge or diverge. If an integral converges, give its exact value.

a. $\displaystyle\int_0^{\infty}(1-x)e^{-x}\,dx$ b. $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin x}{1-\cos x}\,dx$

4. Find the area of the region enclosed by the curves defined by $4x+y^2=0$ and $y=2x+4$.

5. Let $\mathscr{R}$ be the region bounded by the graph of $y=x^3$, the y-axis, and the line defined by $y=8$. Set up, but do not evaluate, integrals representing the volume of the solids obtained by revolving $\mathscr{R}$ about:

a. the x-axis; b. the y-axis; c. the line defined by $x=-2$.

6. Find the length of the curve defined by

$$y=\tfrac{1}{2}\left(e^x+e^{-x}\right),\quad 0\leqslant x\leqslant 1.$$

7. Solve the differential equation

$$\cos^2 x\,\frac{dy}{dx}+y\tan x=0,$$

with the initial condition $y(0)=e$. Express y as a function of x.

8. Consider the sequence $\{a_n\}$ defined by

$$a_n=\frac{3n+\ln n}{2-5n},\qquad \text{for}\quad n\geqslant 1.$$

a. Does the sequence $\{a_n\}$ converge? If so, to what value?

b. Does the series

$$\sum_{n=1}^{\infty} a_n$$

converge? Justify your answer.

9. For each series, determine whether it is convergent or divergent. State which test(s) you are using, and display a proper solution.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n!\,(n+1)!}{(2n)!}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{3+\cos n}{\sqrt{n}}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{\sqrt{n^5+3n^2+7}}$

10. For each series, determine whether it is absolutely convergent, conditionally convergent, or divergent. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{2n+1}{3n+4}\right)^{2n}$ b. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$

11. Find the sum of the series

$$\sum_{n=2}^{\infty}(-1)^n\frac{3}{2^n}.$$

12. Find the radius and interval of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(3n+1)\,4^n}(x+2)^n.$$

13. Let

$$f(x)=\frac{1}{(x+1)^3}.$$

Find the Maclaurin series of f, write the first five non-zero explicitly, and express the series in sigma notation.

1. a. The resolution into partial fractions of the integrand has the form

$$\frac{A}{x+1}+\frac{Bx+C}{4x^2+9}=\frac{7x-6}{(x+1)(4x^2+9)}.$$

Multiplying by $x+1$ and evaluating at -1 gives and $A=-13/13=-1$. Clearing denominators gives

$$A(4x^2+9)+(Bx+C)(x+1)=7x-6.$$

Comparing the coefficients of x^2 gives $4A+B=0$, so $B=4$. Comparing the constant coefficients gives $9A+C=-6$, so $C=3$. Therefore, the integral in question is equal to

$$\begin{aligned}&\int\left\{\frac{-1}{x+1}+\frac{4x}{4x^2+9}+\frac{3}{4x^2+9}\right\}dx\\&=-\log|x+1|+\tfrac12\log(4x^2+9)+\tfrac12\arctan\left(\tfrac23x\right)+C\\&=\tfrac12\log\frac{4x^2+9}{(x+1)^2}+\tfrac12\arctan\left(\tfrac23x\right)+C.\end{aligned}$$

b. If $t=\sec(e^x)$, then $dt=e^x\sec(e^x)\tan(e^x)\,dx$ and $\tan^2(e^x)=t^2-1$; hence,

$$\begin{aligned}\int e^x\sec^3(e^x)\tan^3(e^x)\,dx&=\int t^2(t^2-1)\,dt=\int(t^4-t^2)\,dt\\&=\tfrac15\sec^5(e^x)-\tfrac13\sec^3(e^x)+C.\end{aligned}$$

c. If $y=\sqrt{5+4x-x^2}=\sqrt{9-(x-2)^2}$, then $\frac{dy}{dx}=-(x-2)/y$ and $\int dx/y=\arcsin\left(\frac13(x-2)\right)+C$. So partial integration gives

$$\begin{aligned}\int y\,dx&=(x-2)y+\int\frac{(x-2)^2}{y}\,dx\qquad[\,(x-2)^2=9-y^2\,]\\&=(x-2)y+9\int\frac{dx}{y}-\int y\,dx\\&=\tfrac12(x-2)\sqrt{5+4x-x^2}+\tfrac92\arcsin\left(\tfrac13(x-2)\right)+C.\end{aligned}$$

d. Integrating by parts and then by inspection gives

$$\begin{aligned}\int_0^{\frac16}\arccos(3x)\,dx&=x\arccos(3x)\Big|_0^{\frac16}+3\int_0^{\frac16}\frac{x}{\sqrt{1-9x^2}}\,dx\\&=\tfrac1{18}\pi-\tfrac13\sqrt{1-9x^2}\Big|_0^{\frac16}=\tfrac1{18}\pi-\tfrac13\left(\tfrac12\sqrt3-1\right)\\&=\tfrac1{18}\pi-\tfrac16\sqrt3+\tfrac13.\end{aligned}$$

e. If $t=\sqrt{(x+2)}$, then $x=t^2-2$ and $dx/t=2\,dt$, so the integral is equal to

$$\begin{aligned}2\int(t^2-2+\cos t)\,dt&=\tfrac23\sqrt{(x+2)}^3-4\sqrt{(x+2)}+2\sin\sqrt{(x+2)}+C\\&=\tfrac23(x-4)\sqrt{(x+2)}+2\sin\sqrt{(x+2)}+C.\end{aligned}$$

f. If $t=1/x$, then $dx=-dt/x^2$, and partial integration gives

$$\begin{aligned}\int\frac{1}{x^3}\sin(1/x)\,dx&=-\int t\sin t\,dt=t\cos t-\sin t+C\\&=(1/x)\cos(1/x)-\sin(1/x)+C.\end{aligned}$$

2. a. One application of l'Hôpital's Rule gives

$$\lim_{x\to0^+}\frac{\log(e^x+x)}{x}=\lim_{x\to0^+}\frac{e^x+1}{e^x+x}=2,$$

and therefore, $\lim\limits_{x\to0^+}(e^x+x)^{2/x}=e^4$.

b. If $t=\pi/x$, then a basic trigonometric limit from Calculus I gives

$$\lim_{x\to\infty}x\sin\left(\frac{\pi}{x}\right)=\pi\lim_{t\to0^+}\frac{\sin t}{t}=\pi.$$

c. Combining the logarithms and letting Letting $t=x^{-1/2}$ gives

$$\lim_{x\to\infty}\left\{\log(3x^2+5)-\log(x^2+1)\right\}=\lim_{t\to0^+}\log\frac{3+5t}{1+t}=\log3.$$

3. a. Since $\lim\limits_{x\to\infty}(x-1)e^{-x}=0$ (by l'Hôpital's Rule), partial integration gives

$$\begin{aligned}\int_0^\infty(1-x)e^{-x}\,dx&=\lim_{t\to\infty}(x-1)e^{-x}\Big|_0^t-\int_0^\infty e^{-x}\,dx\\&=-1+\lim_{t\to\infty}e^{-x}\Big|_0^t=-1+1=0.\end{aligned}$$

b. Integrating by inspection gives

$$\int_0^{\frac12\pi}\frac{\sin x}{1-\cos x}\,dx=\lim_{t\to0^+}\log(1-\cos x)\Big|_t^{\frac12\pi}=\infty,$$

so the integral is divergent.

4. The parabola is defined by $4x+y^2=0$, *i.e.*, $4x=-y^2$, and the line is defined by $y=2x+4$, *i.e.*, $4x=2y-8$. The curves intersect where $8-2y=y^2$, or $0=y^2+2y-8=(y+4)(y-2)$. If $-4<y<2$ then the parabola lies to the right of the line, so the area of the region they enclose is equal to

$$\begin{aligned}-\tfrac14\int_{-4}^2(y^2+2y-8)\,dy&=-\tfrac14\left(\tfrac13y^3+y^2-8y\right)\Big|_{-4}^2\\&=-\tfrac14(24-12-48)\\&=9.\end{aligned}$$

5. First observe that the region $\mathscr{R}$ is defined by $x^3\leqslant y\leqslant8$, for $0\leqslant x\leqslant2$.

a. The solid obtained by revolving $\mathscr{R}$ about the x-axis consists of annuli of inner radius x^3, and outer radius 8, for $0\leqslant x\leqslant2$, so its volume is equal to

$$\pi\int_0^2(64-x^6)\,dx.$$

b. The solid obtained by rotating $\mathscr{R}$ about the y-axis consists of concentric cylinders of radius x and height $8-x^3$, for $0\leqslant x\leqslant2$, so its volume is equal to

$$2\pi\int_0^2x(8-x^3)\,dx.$$

c. The solid obtained by revolving $\mathscr{R}$ about the line defined by $x=-2$ can be decomposed into concentric cylinders of radius $x+2$ and height $8-x^3$, for $0\leqslant x\leqslant2$, so its volume is equal to

$$2\pi\int_0^2(x+2)(8-x^3)\,dx.$$

Note: Since $\mathscr{R}$ is also defined by $0\leqslant x\leqslant\sqrt[3]{y}$, for $0\leqslant y\leqslant8$, it is a simple matter to express each volume using y as the variable of integration.

6. Since $y=\frac12(e^x+e^{-x})$ satisfies $1+\left(\frac{dy}{dx}\right)^2=y^2$, and is everywhere positive, the length of the curve in question is equal to

$$\int_0^1y\,dx=\tfrac12(e^x-e^{-x})\Big|_0^1=\tfrac12(e-e^{-1}).$$

7. Separating variables gives

$$\frac1y\frac{dy}{dx}=-\tan x\sec^2x,\qquad\text{and so}\qquad\log|y|=C-\tfrac12\tan^2x,$$

or $y=A\exp(-\frac12\tan^2x)$, for some real number A. Then $y(0)=e$ gives $A=e$, and therefore, $y=e\exp(-\frac12\tan^2x)=\exp(1-\frac12\tan^2x)$.

8. a. Since $\lim\left\{(\log n)/n\right\}=0$ (by the Mean Value Theorem, or l'Hôpital's Rule), it follows (extracting dominant powers) that

$$\lim a_n=\lim\frac{3+(\log n)/n}{2/n-5}=-\tfrac35.$$

b. Since $\lim a_n\neq0$, the Vanishing Condition implies that $\sum a_n$ is divergent.

9. a. If $n\geqslant1$ and

$$a_n=\frac{3^n\,n!\,(n+1)!}{(2n)!},$$

then $0<a_n$ and

$$\lim\frac{a_{n+1}}{a_n}=\lim\frac{3(n+1)(n+2)}{(2n+2)(2n+1)}=\lim\frac{3(1+2/n)}{2(2+1/n)}=\tfrac34<1.$$

Hence, the Ratio Test implies that $\sum a_n$ is convergent.

b. If $n \geqslant 1$, then $1 + \cos n > 0$, which implies that

$$a_n = \frac{3 + \cos n}{\sqrt{n}} > \frac{2}{\sqrt{n}} > 0,$$

Since $\sum n^{-1/2}$ is a divergent p-series ($p = \frac{1}{2} \leqslant 1$), the series $\sum a_n$ is divergent by the Comparison Test.

c. If $n \geqslant 1$, then

$$a_n = \frac{n+1}{\sqrt{n^5 + 3n^2 + 7}} \leqslant \frac{2n}{\sqrt{n^5 + 3n^2 + 7}} < \frac{2n}{\sqrt{n^5}} = \frac{2}{n^{3/2}}.$$

As $\sum n^{-3/2}$ is a convergent p-series ($p = \frac{3}{2} > 1$), the Comparison Test implies that the series $\sum a_n$ is convergent.

10. a. If $n \geqslant 1$ and

$$a_n = \left(\frac{2n+1}{3n+4}\right)^{2n},$$

then $a_n > 0$ and

$$\lim \sqrt[n]{a_n} = \lim \left(\frac{2n+1}{3n+4}\right)^2 = \lim \left(\frac{2+1/n}{3+4/n}\right)^2 = \tfrac{4}{9} < 1.$$

Therefore, the series $\sum (-1)^n a_n$ is absolutely convergent by the Root Test.

b. If $n \geqslant 3$ then $a_n = (\log n) n^{-1/2} > n^{-1/2}$, so the series $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$ ($p = \frac{1}{2} \leqslant 1$) by the Comparision Test. On the other hand, $\lim a_n = 0$ (*e.g.*, by l'Hôpital's Rule) and

$$\frac{d}{dx}\left\{\frac{\log x}{\sqrt{x}}\right\} = \frac{2 - \log x}{2x\sqrt{x}},$$

which is negative if $x > e^2$. So $a_{n+1} < a_n$ if $n \geqslant 8$. Hence, the series $\sum (-1)^n a_n$ is convergent by the Alternating Series Test. Therefore, the series $\sum (-1)^n a_n$ is conditionally convergent.

11. The series in question is a geometric series whose first term is $\frac{3}{4}$ and whose ratio, $-\frac{1}{2}$, has magnitude smaller than 1. Therefore,

$$\sum_{n=2}^{\infty} (-1)^n \frac{3}{2^n} = \frac{3}{4} \cdot \frac{1}{1 - (-1/2)} = \frac{1}{2}.$$

12. If

$$\alpha_n = \frac{(-1)^{n+1}}{(3n+1)\,4^n}(x+2)^n,$$

and $x \neq -2$, then

$$\lim \left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \tfrac{1}{4} \lim \frac{3 + 1/n}{3 + 4/n} |x+2| = \tfrac{1}{4}|x+2|.$$

By Ratio Test, the series $\sum \alpha_n$ is absolutely convergent if $\frac{1}{4}|x+2| < 1$, *i.e.*, $|x+2| < 4$, or $-6 < x < 2$, and divergent if $x < -6$ or $x > 2$. If $x = -6$ and $n \geqslant 1$, then

$$\alpha_n = \frac{-1}{3n+1} < \frac{-1}{3n+n} = -\frac{1}{4n} < 0,$$

in which case $\sum \alpha_n$ diverges with the harmonic series by the Comparison Test. If $x = 2$ and $n \geqslant 0$ then $\alpha_n = (-1)^{n+1} a_n$, where

$$a_n = \frac{1}{3n+1} > \frac{1}{3n+4} = a_{n+1} > 0, \qquad \text{and} \qquad \lim a_n = 0,$$

so the Alternating Series Test implies that $\sum \alpha_n$ is convergent. Hence, the radius of convergence of $\sum \alpha_n$ is 4 and the interval of convergence of $\sum \alpha_n$ is $(-6, 2\,]$.

13. If $|x| < 1$ then

$$\begin{aligned} f(x) = \frac{1}{(1+x)^3} &= \tfrac{1}{2} \frac{d^2}{dx^2}\left\{\frac{1}{1+x}\right\} = \tfrac{1}{2}\frac{d^2}{dx^2} \sum_{n=0}^{\infty} (-1)^n x^n \\ &= \tfrac{1}{2} \sum_{n=2}^{\infty} (-1)^n n(n-1) x^{n-2} \\ &= \sum_{n=0}^{\infty} (-1)^n \tfrac{1}{2}(n+2)(n+1)\, x^n \\ &= 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + \cdots \end{aligned}$$

(The series could also be obtained using the binomial expansion.)