1. Evaluate the following integrals.

a. $\displaystyle\int \frac{x^3+4}{x^2(x+2)^2}\,dx$ b. $\displaystyle\int \frac{\log x}{x^{3/2}}\,dx$ c. $\displaystyle\int x\sec^6(x^2)\,dx$

d. $\displaystyle\int_0^1 x\arctan x\,dx$ e. $\displaystyle\int \frac{dx}{(4x^2+1)^{5/2}}$ f. $\displaystyle\int_0^{\frac{1}{3}\pi} \frac{\log(\sec x+\tan x)}{\cos x}\,dx$

2. Evaluate the following limits.

a. $\displaystyle\lim_{x\to 1^-} \frac{\arccos x}{\sqrt{1-x}}$ b. $\displaystyle\lim_{x\to 0}(e^x-x)^{1/x^2}$

3. Evaluate each improper integral or show that it diverges.

a. $\displaystyle\int_1^2 \frac{dx}{\sqrt{4-x^2}}$ b. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$ c. $\displaystyle\int_0^1 \frac{dx}{\sqrt{(-\log x)^3}}$

4. Find the area of the region enclosed by the curves $y=x^3-12x$ and $y=x^2$.

5. The region $\mathscr{R}=\{(x,y)\colon x^2+y\leqslant 3, x\geqslant 0, y\geqslant 0\}$ is is divided into two parts by the graph of $y=2x$. Let $\mathscr{R}_1$ be the part of the region above the line, and let $\mathscr{R}_2$ be the part of the region below the line. Find the volume of the solid obtained by revolving:

a. $\mathscr{R}_1$ about the y-axis; b. $\mathscr{R}_2$ about the line defined by $x=-1$.

6. Express y as a function of x if $y=4$ when $x=0$ and

$$\sqrt{x^2+9}\,\frac{dy}{dx}=xe^{4-y}.$$

7. A chemical plant discharges toxic solvents into the ground at a rate of 5 tons per year. These solvents do not all stay in the ground: each year, one-tenth of the total quantity of solvents evaporates into the air.

a. Find a formula for the total quantity of solvents in the ground after t years, assuming there are initially none.

b. In the long run, how many tons of solvents accumulate in the ground?

8. Determine whether the sequence $\{a_n\}$ is convergent or divergent. Justify your answer: if the sequence converges, find its limit; otherwise, explain why the sequence diverges.

a. $a_n=\dfrac{\cos(n!)}{5n+1}$ b. $a_n=(-1)^n\dfrac{e^n}{e^n+n}$

9. Given the series

$$\sum_{n=1}^{\infty}\log\left(\frac{2n-1}{2n+5}\right),$$

find an expression for the sum of the first n terms, and use this expression to find the sum of the series, or to show that the series is divergent.

10. Determine whether the series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2n+3}}{5n^2-4n}$ c. $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{\log n}{\sqrt{n}}-\frac{1}{6^n}\right\}$

11. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{2^{n+1}}{n+2^n}\right)^{-n}$ b. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n\sqrt{\log n}}$

12. Find the radius and interval of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{3^n(x+1)^n}{\sqrt{2n+1}}.$$

13. Find the Maclaurin series of $f(x)=(x+1)e^{2x}$.

14. Given that the following series is absolutely convergent,

$$\sum_{n=1}^{\infty}a_n,$$

are the series below convergent or divergent? Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+a_n^2}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{n}$ d. $\displaystyle\sum_{n=1}^{\infty}\sqrt{|a_n a_{n+1}|}$

Solutions

1. a. The resolution into partial fractions of the integrand has the form

$$\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+2}+\frac{d}{(x+2)^2}=\frac{x^3+4}{x^2(x+2)^2},$$

where $b=1$, $d=-1$ are obtained by multiplying and evaluating. Clearing denominators gives

$$ax(x+2)^2+(x+2)^2+cx^2(x+2)-x^2=x^3+4.$$

Comparing the coefficients of x^3 and x gives, respectively, $a+c=1$ and $4a+4=0$. Hence, $a=-1$ and $c=2$, so the integral in question is equal to

$$\int\left\{-\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x+2}-\frac{1}{(x+2)^2}\right\}dx=\log\frac{(x+2)^2}{|x|}-\frac{1}{x}+\frac{1}{x+2}+C$$

$$=\log\frac{(x+2)^2}{|x|}-\frac{2}{x(x+2)}+C.$$

b. Partial integration (integrating the power) gives

$$\int\frac{\log x}{x^{3/2}}\,dx=-\frac{2\log x}{\sqrt{x}}+2\int\frac{dx}{x^{3/2}}=-\frac{2\log x}{\sqrt{x}}-\frac{4}{\sqrt{x}}+C$$

$$=-\frac{2(2+\log x)}{\sqrt{x}}+C.$$

c. If $t=\tan(x^2)$, then $\frac{1}{2}\frac{dt}{dx}=x\sec^2(x^2)$, and $\sec^4(x^2)=(t^2+1)^2=t^4+2t^2+1$, so

$$\int x\sec^6(x^2)\,dx=\tfrac{1}{2}\int(t^4+2t^2+1)\,dt$$

$$=\tfrac{1}{10}\tan^5(x^2)+\tfrac{1}{3}\tan^3(x^2)+\tfrac{1}{2}\tan(x^2)+C.$$

d. Partial integration (integrating the power) gives

$$\int_0^1 x\arctan x\,dx=\tfrac{1}{2}(x^2+1)\arctan x\Big|_0^1-\tfrac{1}{2}\int_0^1 dx=\tfrac{1}{4}\pi-\tfrac{1}{2}.$$

e. If $y=x^{-1}\sqrt{4x^2+1}$, then $y^2=4+x^{-2}$ and $x^{-3}\,dx=-y\,dy$. Then

$$\int\frac{dx}{(4x^2+1)^{5/2}}=\int\frac{x^{-2}}{\left(\frac{\sqrt{4x^2+1}}{x}\right)^5}\cdot x^{-3}\,dx=\int\frac{y^2-4}{y^5}\cdot(-y)\,dy$$

$$=\int\left(-y^{-2}+4y^{-4}\right)dy=\frac{1}{y}-\frac{4}{3y^3}+C=\frac{3y^2-4}{3y^3}+C.$$

Now $3y^2-4=3x^{-2}+8=x^{-2}(3+8x^2)$ and $y^{-3}=x^3(4x^2+1)^{-3/2}$, and hence

$$\int\frac{dx}{(4x^2+1)^{5/2}}=\frac{x(3+8x^2)}{3(4x^2+1)^{3/2}}+C.$$

f. Integrating by inspection gives

$$\int_0^{\frac{1}{3}\pi} \frac{\log(\sec x + \tan x)}{\cos x}\,dx = \tfrac{1}{2}\Big(\log(\sec x + \tan x)\Big)^2\Big|_0^{\frac{1}{3}\pi}$$
$$= \tfrac{1}{2}\Big(\log(2+\sqrt{3})\Big)^2.$$

2. a. As $x \to 1^-$, $t = \arccos x \to 0^+$, and $\sqrt{1-x} = \sqrt{1-\cos t} = \sqrt{2}\sin\left(\tfrac{1}{2}t\right)$. So

$$\lim_{x\to 1^-} \frac{\arccos x}{\sqrt{1-x}} = \frac{2}{\sqrt{2}} \lim_{t\to 0^+} \frac{\frac{1}{2}t}{\sin\left(\frac{1}{2}t\right)} = \sqrt{2},$$

since $(\sin\vartheta)/\vartheta \to 1$ as $\vartheta \to 0$.

b. Since $\lim\limits_{x\to 0}(e^x - x)^{1/(e^x-1-x)} = e$, and (using the Maclaurin expansion of the exponential function)

$$\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} = \lim_{x\to 0}\frac{\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots}{x^2} = \frac{1}{2},$$

it follows that $\lim\limits_{x\to 0}(e^x - x)^{1/x^2} = \sqrt{e}$.

3. a. Using a standard primitive,

$$\int_1^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{\alpha\to 2^-} \arcsin\left(\tfrac{1}{2}x\right)\Big|_1^{\alpha} = \tfrac{1}{2}\pi - \tfrac{1}{6}\pi = \tfrac{1}{3}\pi.$$

b. Since $\frac{d}{dt}\Big(\arctan(e^x)\Big) = e^x/(e^{2x}+1) = (e^x + e^{-x})^{-1}$, it follows that

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \lim_{\lambda\to -\infty} \arctan(e^x)\Big|_{\lambda}^{0} + \lim_{\mu\to\infty} \arctan(e^x)\Big|_0^{\mu} = \tfrac{1}{2}\pi.$$

c. If $0 < x < 1$, then $0 < -\log x < x^{-1}(1-x)$ by the definition of the logarithm; raising these terms to the power $-\frac{3}{2}$, and noting that $\frac{1}{8} < x^{3/2}$ if $\frac{1}{4} < x$, gives

$$0 < \frac{1}{8(1-x)^{3/2}} < \frac{x^{3/2}}{(1-x)^{3/2}} < \frac{1}{\sqrt{(-\log x)^3}}, \qquad \text{provided} \quad \tfrac{1}{4} < x < 1.$$

Hence, the comparison principle implies that the improper integral

$$\int_{\frac{1}{4}}^{1} \frac{dx}{\sqrt{(-\log x)^3}} \qquad \text{diverges with} \qquad \int_{\frac{1}{4}}^{1} \frac{dx}{(1-x)^{3/2}} = \int_0^{\frac{3}{4}} \frac{dt}{t^{3/2}}$$

($p = \frac{3}{2} \geqslant 1$ in the scale of powers at the origin). Therefore, the integral in question diverges.

4. Let $y_1 = x^3 - 12x$ and $y_2 = x^2$; then $y_1 - y_2 = x^3 - x^2 - 12x = x(x+3)(x-4)$, which is zero if $x = -3, 0, 4$, negative if $x < -3$ or $0 < x < 4$ and positive if $-3 < x < 0$ or $x > 4$. The area of the region enclosed by the curves is equal to

$$\int_{-3}^{0} (x^3 - x^2 - 12x)\,dx - \int_0^4 (x^3 - x^2 - 12x)\,dx.$$

The first integral is equal to

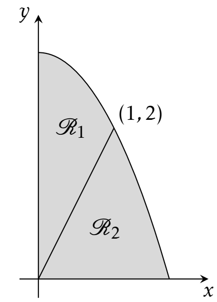
$$\left(\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 6x^2\right)\Big|_{-3}^{0} = -\tfrac{81}{4} - 9 + 54 = \tfrac{99}{4},$$

and the second integral is equal to

$$\left(\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 6x^2\right)\Big|_0^4 = 64 - \tfrac{64}{3} - 96 = -\tfrac{160}{3},$$

so the area of the region enclosed by the curves is equal to $\frac{99}{4} + \frac{160}{3} = \frac{937}{12}$.

5. By inspection, the line defined by $y = 2x$ crosses the region at the origin and at $(1,2)$ (where it meets the graph of $y = 3 - x^2$). Below is a sketch, with $\mathscr{R}_1$, $\mathscr{R}_2$ and the point of intersection in the first quadrant labelled.



a. The solid obtained by revolving $\mathscr{R}_1$ about the y-axis can be decomposed into concentric cylinders of radius x and height $3 - 2x - x^2$, for $0 \leqslant x \leqslant 1$, so its volume is equal to

$$2\pi\int_0^1 (3x - 2x^2 - x^3)\,dx = 2\pi\left(\tfrac{3}{2}x^2 - \tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^1 = 2\pi\left(\tfrac{3}{2} - \tfrac{2}{3} - \tfrac{1}{4}\right) = \tfrac{7}{6}\pi.$$

b. The solid obtained by revolving $\mathscr{R}_2$ about the line defined by $x = -1$ can be decomposed into annuli of inner radius $1 + \frac{1}{2}y$ and outer radius $1 + \sqrt{3-y}$, for $0 \leqslant y \leqslant 2$, so its volume is equal to

$$\pi\int_0^2 \Big((1+\sqrt{3-y})^2 - (1+\tfrac{1}{2}y)^2\Big)\,dy = \pi\int_0^2 \Big(3 + 2\sqrt{3-y} - 2y - \tfrac{1}{4}y^2\Big)\,dy$$
$$= \pi\Big(3y - \tfrac{4}{3}(3-y)^{3/2} - y^2 - \tfrac{1}{12}y^3\Big)\Big|_0^2 = \pi\Big(6 - \tfrac{4}{3} - 4 - \tfrac{2}{3} + 4\sqrt{3}\Big)$$
$$= 4\pi\sqrt{3}.$$

6. Separating variables and integrating gives

$$e^{y-4}\,\frac{dy}{dx}\,\frac{x}{\sqrt{x^2+9}}, \qquad \text{and hence} \qquad e^{y-4} = \sqrt{x^2+9} + \alpha,$$

for some real number α. Since $y = 4$ if $x = 0$, it follows that $\alpha = -2$, and therefore $y = 4 + \log\Big(\sqrt{x^2+9} - 2\Big)$.

7. If q denotes the quantity (in tons) of solvent in the ground at time t then it is given that

$$\frac{dq}{dt} = 5 - \tfrac{1}{10}q, \qquad \text{or} \qquad \frac{d}{dt}(q - 50) = -\tfrac{1}{10}(q - 50),$$

which is plainly an exponential equation, so there is a real number A such that $q = 50 + Ae^{-t/10}$. Since $q = 0$ when $t = 0$, it follows that $A = -50$, or $q = 50(1 - e^{-t/10})$. As $t \to \infty$, the quantity q approaches fifty tons.

8. a. Since $|\cos(n!)/(5n+1)| < 1/n$ if $n \geqslant 1$, $\lim a_n = 0$ by definition.

b. Since $\lim|1/a_n| = \lim(1 + n/e^n) = 0$ (by the basic arithmetic of logarithms), it follows that $\lim a_{2n} = 1$ and $\lim a_{2n+1} = -1$. Therefore, $\{a_n\}$ is divergent.

9. If

$$a_n = \log\left(\frac{2n-1}{2n+5}\right) \qquad \text{and} \qquad b_n = \log\Big((2n-1)(2n+1)(2n+3)\Big),$$

then $a_n = b_n - b_{n+1}$, and hence

$$a_1 + a_2 + \cdots + a_n = b_1 - b_{n+1} = \log 15 - \log\Big((2n+1)(2n+3)(2n+5)\Big),$$

for $n \geqslant 1$. Therefore, $\lim\limits_{n\to\infty}(a_1 + a_2 + a_3 + \cdots + a_n) = -\infty$; in other words, the series $\sum a_n$ is divergent.

10. a. Since $\sin\vartheta > \frac{1}{2}\vartheta$ if $0 < \vartheta < \frac{1}{3}\pi$, $a_n = n\sin(1/n) > n/(2n) > \frac{1}{2}$ if $n \geqslant 1$, so the vanishing condition implies that $\sum a_n$ is divergent.

Note. — The fact that $\lim\{n\sin(1/n)\} = \lim\limits_{t\to 0}\frac{\sin t}{t} = 1$ could be used instead.

b. If $n \geqslant 4$ then

$$0 < a_n = \frac{\sqrt{2n+3}}{5n^2-4n} < \frac{\sqrt{4n}}{4n^2} = \frac{1}{2n^{3/2}},$$

so $\sum a_n$ converges with the p-series $\sum n^{-3/2}$ ($\frac{3}{2} > 1$) by the comparison test.

Note. — The limit comparison test could be applied instead, for

$$\lim \frac{a_n}{n^{-3/2}} = \lim \frac{\sqrt{2+3n^{-1}}}{5-4n^{-1}} = \tfrac{1}{5}\sqrt{2}.$$

c. If $n \geqslant 8$ then $2 < \log n < \sqrt{n}$ and $6^n > e^n > n > \sqrt{n}$, so

$$a_n = \frac{\log n}{\sqrt{n}} - \frac{1}{6^n} > \frac{\log n - 1}{\sqrt{n}} > \frac{1}{\sqrt{n}},$$

so $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$ ($\frac{1}{2} \leqslant 1$) by the comparison test.

Note. — Another argument could be given along the following lines. Since $n^{-1/2}\log n > n^{-1/2}$ if $n \geqslant 3$, the series $\sum n^{-1/2}\log n$ diverges with $\sum n^{-1/2}$ by the comparison test. Also, $\sum 6^{-n}$ is a convergent geometric series (whose ratio is $\frac{1}{6}$). Thus, $\sum\left\{n^{-1/2}\log n - 6^{-n}\right\}$, being the termwise difference of a divergent series and a convergent series, is divergent.

11. a. If $a_n = \left(\frac{1}{2}\right)^n\left(n2^{-n}+1\right)^n$, then $\lim \sqrt[n]{a_n} = \frac{1}{2}\lim\left(n2^{-n}+1\right) = \frac{1}{2} < 1$, so the root test implies that $\sum(-1)^n a_n$ is absolutely convergent.

b. Since $0 < \log(1+1/n) < 1/n$ if $n \geqslant 1$, rationalizing the numerator gives

$$0 < 2\sqrt{\log(n+1)} - 2\sqrt{\log n} = \frac{2\log(1+1/n)}{\sqrt{\log(n+1)}+\sqrt{\log n}} < \frac{1}{n\sqrt{\log n}} = a_n,$$

so $a_2 + a_3 + \cdots + a_{n+1} > 2\sqrt{\log(n+2)} - 2\sqrt{\log 2}$ if $n \geqslant 2$, which implies that $\sum a_n$ is divergent. However, since $a_{n+1} < a_n$ if $n \geqslant 2$, and $\lim a_n = 0$, the alternating series test implies that $\sum(-1)^{n-1}a_n$ is convergent. Therefore, the series $\sum(-1)^{n-1}a_n$ is conditionally convergent.

Note. — It could be observed that an alternating logarithmic p-series with $p \leqslant 1$ is conditionally convergent. Also, the divergence of $\sum a_n$ is explained immediately by condensing: $2^n a_n = 2^n\left(2^n\sqrt{\log(2^n)}\right)^{-1} = (n\log 2)^{-1/2}$ is a non-zero multiple of the general term of a divergent p-series ($p = \frac{1}{2} \leqslant 1$).

12. If $x \neq -1$ and

$$u_n = \frac{3^n(x+1)^n}{\sqrt{2n+1}}, \quad \text{then} \quad \lim\left|\frac{u_{n+1}}{u_n}\right| = 3|x+1|\lim\sqrt{\frac{2+1/n}{2+3/n}} = 3|x+1|,$$

so the ratio test implies that $\sum u_n$ is absolutely convergent if $|x+1| < \frac{1}{3}$, *i.e.*, $-\frac{4}{3} < x < -\frac{2}{3}$, and divergent if $x < -\frac{4}{3}$ or $x > -\frac{2}{3}$. Let $a_n = (2n+1)^{-1/2}$. If $x = -\frac{2}{3}$ and $n \geqslant 1$, then $u_n = a_n > \frac{1}{2}n^{-1/2}$, so the series $\sum u_n$ diverges with the p-series $\sum n^{-1/2}$ ($\frac{1}{2} \leqslant 1$) by the comparison test. If $x = -\frac{4}{3}$, then $\sum u_n = \sum(-1)^n a_n$ converges by the alternating series test, since $a_n > a_{n+1}$ if $n \geqslant 0$ and $\lim a_n = 0$. Therefore, the series $\sum u_n$ has radius of convergence $\frac{1}{3}$ and interval of convergence $\left[-\frac{4}{3}, -\frac{2}{3}\right)$.

13. Using the Maclaurin expansion of the exponential function gives

$$\begin{aligned}(x+1)e^{2x} &= \sum_{k=0}^{\infty}\frac{2^k}{k!}x^k(1+x) = 1 + \sum_{k=1}^{\infty}\left\{\frac{2^k}{k!} + \frac{2^{k-1}}{(k-1)!}\right\}x^k \\ &= \sum_{k=0}^{\infty}\frac{2^{k-1}(k+2)}{k!}x^k \\ &= 1 + 3x + 4x^2 + \tfrac{10}{3}x^3 + 2x^4 + \tfrac{7}{15}x^5 + \tfrac{16}{45}x^6 + \tfrac{4}{35}x^7 + \cdots,\end{aligned}$$

which is valid for all real values of x.

14. a. Since $|(-1)^n a_n| = |a_n|$, $\sum(-1)^n a_n$ is (absolutely) convergent.

b. Since $b_n = 1/(1+a_n^2) \to 1/(1+0) = 1$ the vanishing condition implies that the series $\sum b_n$ is divergent.

c. If $n \geqslant 1$, then $0 \leqslant |n^{-1}a_n| \leqslant |a_n|$. Since $\sum|a_n|$ is convergent, the comparison test implies that $\sum\left\{n^{-1}a_n\right\}$ is absolutely convergent.

d. Since

$$0 \leqslant (\sqrt{|a_n|} - \sqrt{|a_{n+1}|})^2 = |a_n| + |a_{n+1}| - 2\sqrt{|a_n||a_{n+1}|},$$

or equivalently,

$$2\sqrt{|a_n||a_{n+1}|} \leqslant |a_n| + |a_{n+1}|,$$

and the series $\sum\{|a_n| + |a_{n+1}|\} = \sum|a_n| + \sum|a_{n+1}|$ is (absolutely) convergent, the comparison test implies that the series $\sum\sqrt{|a_n||a_{n+1}|}$ is convergent.