1. Evaluate the following integrals.

a. $\displaystyle\int \frac{\tan^5(\log x)\sec^3(\log x)}{x}\,dx$ b. $\displaystyle\int x\arcsin x\,dx$ c. $\displaystyle\int \frac{x^2}{\sqrt{4x^2-9}}\,dx$

d. $\displaystyle\int \frac{8x^2+4x+5}{(x+1)^2(2x-1)}\,dx$ e. $\displaystyle\int_5^6 \frac{dx}{\sqrt{-x^2+10x-21}}$ f. $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}+25}}\,dx$

g. $\displaystyle\int \sqrt{x}\,e^{\sqrt{x}}\,dx$ h. $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x^2\arctan x+\arctan x}$

2. Given $f(-2)=-3$, $f'(-2)=5$, $f(1)=3$ and $f'(1)=2$, evaluate

$$\int_{-2}^{1} x f''(x)\,dx.$$

3. Evaluate the following limits.

a. $\displaystyle\lim_{x\to\pi}\frac{\sin^2(3x)}{1+\cos x}$ b. $\displaystyle\lim_{x\to\infty}\left\{x(e^{3/x}-1)\right\}$ c. $\displaystyle\lim_{x\to 0^+}(\cos x)^{1/x^2}$

4. Determine whether each improper integral is convergent. If an integral converges, evaluate it.

a. $\displaystyle\int_1^{\infty}\frac{1-\log x}{x^2}\,dx$ b. $\displaystyle\int_0^3\frac{dx}{x^2-2x+1}$

5. Solve the differential equation

$$\cos^2(x)\,\frac{dy}{dx}=e^{-y}\sin(x); \qquad y(0)=0.$$

6. Sketch the region enclosed by the graphs of

$$y=x^2+1 \qquad \text{and} \qquad y=2x+4,$$

and find its area.

7. Let $\mathscr{R}$ be the region bounded by the graphs of

$$y=1+\cos x, \qquad y=1, \qquad x=0 \qquad \text{and} \qquad x=\tfrac{1}{2}\pi.$$

Set up, but do not evaluate, an integral which is equal to the volume of the solid obtained by revolving $\mathscr{R}$ about: a. the y-axis; b. the x-axis; c. the line defined by $x=3$; d. the line defined by $y=3$;

8. Let $a_n=n\sin(1/n)$, for $n\geqslant 1$.

a. Is $\{a_n\}$ convergent? If so, find its limit. If not, explain why not.

b. Is $\displaystyle\sum_{n=1}^{\infty}a_n$ convergent? Justify your answer.

9. You are given that $a_1+a_2+\cdots+a_n=\dfrac{n}{n+2}$, for $n\geqslant 1$.

a. Evaluate $\displaystyle\sum_{n=1}^{\infty}a_n$. b. Find a_n, for $n\geqslant 1$.

10. Determine whether each series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{2n}{3n+2}-\frac{1}{n\sqrt{n}}\right\}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\,7^n}$ c. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-n}{2n+1}\right)^{3n}$ d. $\displaystyle\sum_{n=1}^{\infty}\frac{5\sqrt{n}}{(n+1)^2}$

11. Determine whether each series is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{\ln(2n)}$ b. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n+1)!}{n^2\,7^n}$

12. Find the radius and interval of convergence of the power series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3^{2k}}{3k+1}(x-1)^k.$$

13. Find the Maclaurin series of $f(x)=\dfrac{x}{x+1}$.

Solutions

1. a. If $s=\sec(\log x)$, then $\frac{ds}{dx}=x^{-1}\sec(\log x)\tan(\log x)$, and

$$\tan^4(\log x)\sec^2(\log x)=(s^2-1)^2s^2=s^6-2s^4+s^2.$$

So the integral in question is equal to

$$\int(s^6-2s^4+s^2)\,ds=\tfrac{1}{7}\sec^7(\log x)-\tfrac{2}{5}\sec^5(\log x)+\tfrac{1}{3}\sec^3(\log x)+a.$$

b. If $\vartheta=\arcsin x$, then $x\frac{dx}{d\vartheta}=\sin\vartheta\cos\vartheta=\frac{1}{2}\sin(2\vartheta)$, and partial integration gives

$$\int x\arcsin x\,dx=\tfrac{1}{2}\int\vartheta\sin(2\vartheta)\,d\vartheta=-\tfrac{1}{4}\vartheta\cos(2\vartheta)+\tfrac{1}{8}\sin(2\vartheta)+b.$$

Since $\sin(2\vartheta)=2\sin\vartheta\cos\vartheta=2x\sqrt{1-x^2}$ and $\cos(2\vartheta)=1-2\sin^2(\vartheta)=1-2x^2$, it follows that

$$\int x\arcsin x\,dx=\tfrac{1}{4}x\sqrt{1-x^2}+\tfrac{1}{4}(2x^2-1)\arcsin x+b.$$

c. Partial integration and revising the remaining integral gives

$$\int x\cdot\frac{x}{\sqrt{4x^2-9}}\,dx=\tfrac{1}{4}x\sqrt{4x^2-9}-\tfrac{1}{4}\int\sqrt{4x^2-9}\,dx$$
$$=\tfrac{1}{4}x\sqrt{4x^2-9}-\int\frac{x^2}{\sqrt{4x^2-9}}\,dx+\tfrac{9}{4}\int\frac{dx}{\sqrt{4x^2-9}}.$$

Solving for the integral in question then gives

$$\int\frac{x^2}{\sqrt{4x^2-9}}\,dx=\tfrac{1}{8}x\sqrt{4x^2-9}+\tfrac{9}{16}\log\left|2x+\sqrt{4x^2-9}\right|+c.$$

d. The resolution into partial fractions of the integrand is

$$\frac{a}{x+1}-\frac{3}{(x+1)^2}+\frac{4}{2x-1}=\frac{8x^2+4x+5}{(x+1)^2(2x-1)},$$

where the second and third coefficients are obtained by multiplying and evaluating. Comparing the quadratic coefficients then gives $a=2$. Therefore,

$$\int\frac{8x^2+4x+5}{(x+1)^2(2x-1)}\,dx=\int\left(\frac{2}{x+1}-\frac{3}{(x+1)^2}+\frac{4}{2x-1}\right)dx$$
$$=2\log|(x+1)(2x-1)|+\frac{3}{x+1}+d.$$

e. Since $-x^2+10x-21=4-(x-5)^2$, the integral in question is equal to

$$\int_5^6\frac{dx}{\sqrt{4-(x-5)^2}}=\arcsin\left(\tfrac{1}{2}(x-5)\right)\Big|_5^6=\tfrac{1}{6}\pi.$$

f. If $t=e^x$, then $dt=e^x\,dx$, and hence

$$\int\frac{e^x}{\sqrt{e^{2x}+25}}\,dx=\int\frac{dt}{\sqrt{t^2+25}}=\log\left(e^x+\sqrt{e^{2x}+25}\right)+f.$$

g. If $t=\sqrt{x}$ then $dx=2t\,dt$, and repeated partial integration gives

$$\int\sqrt{x}\,e^{\sqrt{x}}\,dx=2\int t^2e^t\,dt=2e^t(t^2-2t+2)+g=2e^{\sqrt{x}}(x-2\sqrt{x}+2)+g.$$

h. Since $x^2 \arctan x + \arctan x = (x^2+1)\arctan x$, it the integral in question is equal to

$$\int_1^{\sqrt{3}} \frac{1}{\arctan x} \cdot \frac{dx}{x^2+1} = \log(\arctan x)\Big|_1^{\sqrt{3}} = \log\left(\tfrac{1}{3}\pi\right) - \log\left(\tfrac{1}{4}\pi\right) = \log\left(\tfrac{4}{3}\right).$$

2. As $f(-2) = -3$, $f'(-2) = 5$, $f(1) = 3$ and $f'(1) = 2$, partial integration gives

$$\int_{-2}^{1} x f''(x)\,dx = x f'(x)\Big|_{-2}^{1} - \int_{-2}^{1} f'(x)\,dx = (1)(2) - (-2)(5) - (3-(-3)) = 6.$$

3. a. If $t = \pi - x$, then

$$1 + \cos x = 1 - \cos t = 2\sin^2\left(\tfrac{1}{2}t\right) \qquad \text{and} \qquad \sin^2(3x) = \sin^2(3t).$$

Hence,

$$\lim_{x\to\pi} \frac{\sin^2(3x)}{1+\cos x} = 18 \lim_{t\to 0} \left\{ \frac{\sin^2(3t)}{(3t)^2} \cdot \frac{\left(\frac{1}{2}t\right)^2}{\sin^2\left(\frac{1}{2}t\right)} \right\} = 18.$$

Alternatively, using Maclaurin expansions of the sine and cosine functions,

$$\frac{\sin^2(3x)}{1+\cos x} = \frac{\sin^2(3t)}{1-\cos t} = \frac{\left(3t - \frac{1}{6}(3t)^3 + \cdots\right)^2}{\frac{1}{2}t^2 - \frac{1}{24}t^4 - \ldots} \to 18, \quad \text{as } x \to \pi \ (\textit{i.e.}, t \to 0).$$

b. Letting $t = 3/x$ and using the definition of the derivative yields

$$\lim_{x\to\infty} \left\{ x(e^{3/x} - 1) \right\} = 3 \lim_{t\to 0^+} \frac{e^t - 1}{t} = 3.$$

c. If $t = -2\sin^2\left(\frac{1}{2}x\right)$, then

$$\lim_{t\to 0^-} \frac{t}{x^2} = -\frac{1}{2} \lim_{x\to 0} \left(\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x} \right)^2 = -\frac{1}{2}.$$

Since $\cos x = 1 - 2\sin^2\left(\frac{1}{2}x\right) = 1 + t$, it follows that

$$\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim_{t\to 0^-} \left((1+t)^{1/t}\right)^{t/x^2} = \sqrt{e^{-1}}.$$

4. a. Partial integration, integrating the power, gives

$$\int_1^{\infty} \frac{1 - \log x}{x^2}\,dx = \lim_{t\to\infty} -\frac{1-\log x}{x}\Bigg|_1^t - \int_1^{\infty} \frac{dx}{x^2} = 1 + \lim_{t\to\infty} \frac{1}{x}\Bigg|_1^t = 0.$$

b. If $t = 1 - x$ then

$$\int_0^1 \frac{dx}{x^2 - 2x + 1} = \int_0^1 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dt}{t^2}$$

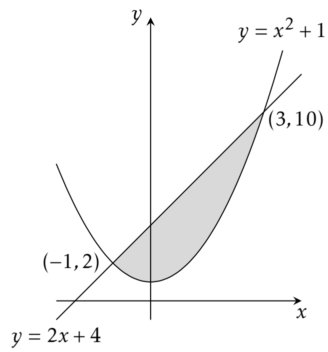
is divergent ($p = 2 \geqslant 1$ in the scale of powers at the origin). Therefore, the integral in question is divergent.

5. Separating variables gives

$$e^y \frac{dy}{dx} = \frac{\sin(x)}{\cos^2(x)}, \quad \text{and hence} \quad e^y = \frac{1}{\cos(x)}, \quad \text{or} \quad y = -\log(\cos x),$$

since $y = 0$ if $x = 0$ (which solution is valid if $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$).

6. Follows a sketch of the region (not to scale—the x-axis is dilated by a factor of 2).



The curves meet where

$$x^2 + 1 = 2x + 4,$$

i.e.,

$$x^2 - 2x - 3 = 0,$$

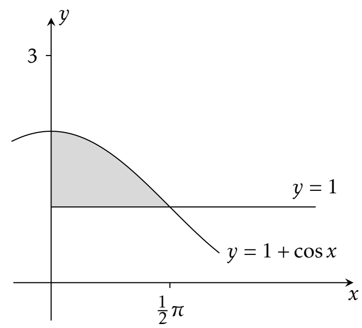
or

$$(x+1)(x-3) = 0.$$

Since the line is above the parabola if $-1 < x < 3$, the area of the region is

$$\int_{-1}^{3} (3 + 2x - x^2)\,dx = \left(3x + x^2 - \tfrac{1}{3}x^3\right)\Big|_{-1}^{3} = 12 + 8 - \tfrac{28}{3} = \tfrac{32}{3}.$$

7. The region $\mathscr{R}$ is sketched below.



a. The solid obtained by revolving $\mathscr{R}$ about the y-axis can be decomposed into concentric cylinders of radius x and height $\cos x$, for $0 \leqslant x \leqslant \frac{1}{2}\pi$, so its volume is equal to

$$2\pi \int_0^{\frac{1}{2}\pi} x\cos(x)\,dx.$$

b. The solid obtained by revolving $\mathscr{R}$ about the x-axis can be decomposed into annuli of inner radius 1 and outer radius $1 + \cos(x)$, for $0 \leqslant x \leqslant \frac{1}{2}\pi$, so its volume is equal to

$$\pi \int_0^{\frac{1}{2}\pi} \{(1+\cos(x))^2 - 1\}\,dx = \pi \int_0^{\frac{1}{2}\pi} \{\cos^2(x) + 2\cos(x)\}\,dx.$$

c. The solid obtained by revolving $\mathscr{R}$ about the line defined by $x = 3$ consists of concentric cylindrical shells of radius $3 - x$ and height $\cos x$, for $0 \leqslant x \leqslant \frac{1}{2}\pi$, so its volume is equal to

$$2\pi \int_0^{\frac{1}{2}\pi} (3-x)\cos(x)\,dx.$$

d. The solid obtained by revolving $\mathscr{R}$ about the line defined by $y = 3$ can be decomposed into annuli of inner radius $2 - \cos(x)$ and outer radius 2, for $0 \leqslant x \leqslant \frac{1}{2}\pi$, so its volume is equal to

$$\pi \int_0^{\frac{1}{2}\pi} \{2^2 - (2-\cos(x))^2\}\,dx = \pi \int_0^{\frac{1}{2}\pi} \{4\cos(x) - \cos^2(x)\}\,dx.$$

8. a. The sequence $\{a_n\}$ converges to 1, since

$$\lim a_n = \lim\{n\sin(1/n)\} = \lim_{t\to 0}\frac{\sin t}{t} = 1.$$

b. Since $\lim a_n \neq 0$, the series $\sum a_n$ is divergent by the vanishing condition.

9. a. By the orthodox definition of the sum of a series,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}(a_1 + \cdots + a_n) = \lim_{n\to\infty}\frac{n}{n+2} = \lim_{n\to\infty}\left\{1 - \frac{2}{n+2}\right\} = 1.$$

b. If $n \geqslant 2$, then

$$a_n = (a_1 + \cdots + a_n) - (a_1 + \cdots + a_{n-1}) = -\frac{2}{n+2} + \frac{2}{n+1} = \frac{2}{(n+1)(n+2)}.$$

In fact, the displayed expression also gives the correct value $a_1 = \frac{1}{3}$.

10. a. Since

$$\lim\left\{\frac{2n}{3n+2} - \frac{1}{n\sqrt{n}}\right\} = \tfrac{2}{3} \neq 0, \qquad \text{the series} \qquad \sum_{n=1}^{\infty}\left\{\frac{2n}{3n+2} - \frac{1}{n\sqrt{n}}\right\}$$

is divergent by the vanishing condition.

b. If $n \geqslant 1$, then $0 < (n7^n)^{-1} \leqslant 7^{-n}$, and $\sum 7^{-n}$ is a convergent geometric series ($|r| = \frac{1}{7} < 1$), so the comparison test implies that the series $\sum (n7^n)^{-1}$ is convergent. (The ratio test could be used as well.)

c. If

$$a_n = \left(\frac{-n}{2n+1}\right)^{3n}, \qquad \text{then} \qquad \lim \sqrt[n]{|a_n|} = \lim\left(\frac{n}{2n+1}\right)^3 = \tfrac{1}{8},$$

so the root test implies that the series $\sum a_n$ is (absolutely) convergent.

d. If

$$a_n = \frac{5\sqrt{n}}{(n+1)^2}, \qquad \text{then} \qquad 0 < a_n < \frac{5\sqrt{n}}{n^2} = \frac{5}{n^{3/2}}, \qquad \text{if} \quad n \geqslant 1.$$

Since $\sum n^{-3/2}$ is a convergent p-series, the comparison test implies that the series $\sum a_n$ is convergent. (The limit comparison test could be used as well.)

11. a. If $n \geqslant 1$, then $0 \leqslant \log n \leqslant n - 1$; hence, $0 < \log(2n) = \log 2 + \log n < n$. Therefore, if

$$a_n = \frac{1}{\log(2n)} \qquad \text{and} \quad n \geqslant 1, \quad \text{then} \qquad 0 < \frac{1}{n} < a_n,$$

so the comparison test implies that $\sum a_n$ diverges with the harmonic series. On the other hand, $\log(2(n+1)) > \log(2n) > 0$ if $n \geqslant 1$, and $\lim \log(2n) = \infty$, which implies that $0 < a_{n+1} < a_n$ if $n \geqslant 1$, and $\lim a_n = 0$. As $\cos(\pi n) = (-1)^n$, the series $\sum \cos(\pi n)a_n$ is convergent by the alternating series test. Thus, $\sum \cos(\pi n)a_n$ is a conditionally convergent series.

b. If $n \geqslant 5$, then

$$a_n = \frac{(2n+1)!}{n^2\, 7^n} = 1\cdot 2\cdots(n-1)\cdot\frac{n(n+1)}{n^2}\cdot\frac{(n+2)(n+3)\cdots(2n+1)}{7^n} > 24,$$

so the vanishing condition implies that $\sum (-1)^{n+1}a_n$ is divergent. (The ratio test could be used as well.)

12. If

$$\beta_k = (-1)^{k+1}\frac{3^{2k}}{3k+1}(x-1)^k,$$

and $x \neq 1$, then

$$\lim\left|\frac{\beta_{k+1}}{\beta_k}\right| = \lim\left|\frac{3k+4}{3^{2k+2}(x-1)^{k+1}}\cdot\frac{3^{2k}(x-1)^k}{3k+1}\right|$$
$$= 9|x-1|\lim\frac{3+4k^{-1}}{3+k^{-1}} = 9|x-1|,$$

so the ratio test implies that $\sum \beta_k$ is absolutely convergent if $|x-1| < \frac{1}{9}$, *i.e.*, $\frac{8}{9} < x < \frac{10}{9}$, and divergent if $x < \frac{8}{9}$ or $x > \frac{10}{9}$. Now,

$$\beta_k = \frac{-1}{3k+1}, \quad \text{if} \ \ x = \tfrac{8}{9}, \qquad \text{and} \qquad \beta_k = \frac{(-1)^{k+1}}{3k+1}, \quad \text{if} \ \ x = \tfrac{10}{9}.$$

Let $b_k = (3k+1)^{-1}$. Since $0 < \frac{1}{4}k^{-1} \leqslant b_k$ if $k \geqslant 1$, the series $\sum b_k$ diverges with the harmonic series by the comparison test; hence, $\sum \beta_k$ diverges if $x = \frac{8}{9}$. On the other hand, $0 < b_{k+1} < b_k$ if $k \geqslant 1$ and $\lim b_k = 0$, so the alternating series test implies that $\sum \beta_k$ converges if $x = \frac{10}{9}$. Therefore, the power series $\sum \beta_k$ has radius of convergence $\frac{1}{9}$ and interval of convergence $\left(\frac{8}{9}, \frac{10}{9}\right]$.

13. Using a standard geometric series gives

$$\frac{x}{x+1} = x\cdot\frac{1}{1-(-x)} = x\sum_{j=0}^{\infty}(-x)^j = \sum_{j=0}^{\infty}(-1)^j x^{j+1} = \sum_{j=1}^{\infty}(-1)^{j-1}x^j,$$

provided $-1 < x < 1$.