1. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{x^2 - x + 13}{(x+1)(x^2+4)}\,dx$ b. $\displaystyle\int e^{-2x}\sin(3x)\,dx$ c. $\displaystyle\int_3^4 \frac{\tan^3(\pi/x)}{x^2}\,dx$

d. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-1}}$ e. $\displaystyle\int \frac{dx}{x\sqrt{2x-9}}$ f. $\displaystyle\int x^5 e^{-x^2}\,dx$

2. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to\infty} \frac{\tan^{-1}(2x) - \frac{1}{2}\pi}{\sin(1/x)}$ b. $\displaystyle\lim_{x\to 0}\left(1+\sin^2(x)\right)^{4/x^2}$

3. Evaluate each of the following improper integrals.

a. $\displaystyle\int_{-1}^{0} \frac{e^{1/x}}{x^2}\,dx$ b. $\displaystyle\int_{e^2}^{\infty} \frac{\log x}{(x\log x - x)^2}\,dx$

4. Find the length of the curve defined by $y = \frac{1}{2}x^2 + \frac{1}{4}\log(x)$, $1 \leqslant x \leqslant 2$.

5. Let $\mathscr{R}$ be the region bounded by the graphs of $y = \sqrt{2x}$ and $y = \frac{1}{2}x^2$.
a. Compute the the area of $\mathscr{R}$.
b. Write an integral which is equal to the volume of the solid obtained by revolving $\mathscr{R}$ about: i. the x-axis; ii. the line $x = -2$; iii. the line $y = 3$.

6. Solve the differential equation

$$xy^2 + x + (x^2y - y)\frac{dy}{dx} = 0,$$

subject to the initial condition $y(0) = -2$.

7. A tank contains 9 kilograms of salt dissolved in 100 litres of water. Pure water enters the tank at a rate of 15 litres per minute. The solution is kept thoroughly mixed, and drains from the tank at a rate of 10 litres per minute. How much salt is in the tank after ten minutes?

8. Consider a sequence $\{a_n\}$ which begins as indicated below.

$$\left\{\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{4}{2}, \tfrac{8}{6}, \tfrac{16}{24}, \tfrac{32}{120}, \tfrac{64}{720}, \ldots\right\}$$

Assuming that the pattern continues, give a formula for a_n such that:
a. $\lim a_n = 0$; b. $\lim a_n = \infty$; c. $\{a_n\}$ oscillates and diverges.

9. Determine whether each series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{3/n^2}}{n}$ b. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{3k\sqrt{\log k} + 5k\sqrt[3]{(\log k)^2} - 7\log k}$

c. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n-1}{25n+1}\right)^{n/2}$ d. $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(3/n)}{2n+1}$

10. Determine whether each series is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{e^n + 1 - n^2}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n + 2\sqrt{n+1}}}$

c. $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{(\log k)^2}{\sqrt[4]{k}}$ d. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(2n)^{2n}}$

11. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin(1/\sqrt{n})}{5^n\sqrt{n+1}}(x+2)^n.$$

12. Compute the Maclaurin series of

$$f(x) = \frac{1}{(1-2x)^2}.$$

Express the series using summation notation, and give the first five terms explicitly and in simplified form.

13. a. Given that

$$\int f(x)\,dx = x\arccos(5x) - \tfrac{1}{5}\sqrt{1-25x^2} + C,$$

find and simplify $f(x)$.

b. Show that the area under the curve defined by $y = e^{\sqrt{x}}$ on the interval $[0, 1]$ is equal to the area under the curve defined by $y = e^{\sin x}\sin(2x)$ on the interval $[0, \frac{1}{2}\pi]$.

c. Determine whether each statement is true or false. Justify your answers.

i. If $a_n \neq 0$ for $n \geqslant 1$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

ii. If $a_n \neq 0$ for $n \geqslant 1$ and $\lim(n^2 a_n) = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

iii. If $a_n \neq 1$ for $n \geqslant 1$ and $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} \frac{a_n\cos(n)}{1-a_n}$ converges.

Solutions

1. a. The resolution into partial fractions of the integrand is
$$\frac{x^2-x+13}{(x+1)(x^2+4)} = \frac{3}{x+1} + \frac{-2x+1}{x^2+4},$$
where the first coefficient is obtained by multiplying and evaluating, and $-2x+1$ is obtained by comparing quadratic and constant coefficients. Thus,
$$\int \frac{x^2-x+13}{(x+1)(x^2+4)}\,dx = \log\left|\frac{(x+1)^3}{x^2+4}\right| + \tfrac{1}{2}\arctan\left(\tfrac{1}{2}x\right) + C.$$
b. Repeated partial integration, integrating the exponential function and differentiating the trigonometric function at each stage, gives
$$\int e^{-2x}\sin(3x)\,dx = -\tfrac{1}{2}e^{-2x}\sin(3x) - \tfrac{3}{4}e^{-2x}\cos(3x) - \tfrac{9}{4}\int e^{-2x}\sin(3x)\,dx.$$
Therefore,
$$\tfrac{13}{4}\int e^{-2x}\sin(3x)\,dx = -\tfrac{1}{4}e^{-2x}(2\sin(3x)+3\cos(3x)) + C,$$
or
$$\int e^{-2x}\sin(3x)\,dx = -\tfrac{1}{13}e^{-2x}(2\sin(3x)+3\cos(3x)) + C.$$
c. If $s=\sec(\pi/x)$, then $\tan^2(\pi/x) = s^2-1$ and
$$ds = -\pi\frac{\sec(\pi/x)\tan(\pi/x)}{x^2}\,dx, \qquad \text{or} \qquad -\frac{ds}{\pi s} = \frac{\tan(\pi/x)}{x^2}\,dx.$$
Also, $s=2$ if $x=3$ and $s=\sqrt{2}$ if $x=4$, so
$$\int_3^4 \frac{\tan^3(\pi/x)}{x^2}\,dx = \frac{1}{\pi}\int_{\sqrt{2}}^{2}\frac{s^2-1}{s}\,ds = \frac{1}{\pi}\left(\tfrac{1}{2}s^2 - \log s\right)\Big|_{\sqrt{2}}^{2} = \frac{1}{\pi}\left(1-\tfrac{1}{2}\log 2\right).$$
d. If $y = x^{-1}\sqrt{x^2-1}$, then $x^{-2} = 1-y^2$, and $x^{-3}\,dx = y\,dy$, or $dy = (x^3y)^{-1}\,dx$. Therefore,
$$\int\frac{dx}{x^4\sqrt{x^2-1}} = \int\frac{dx}{x^5y} = \int(1-y^2)\,dy = y - \tfrac{1}{3}y^3 + C = \tfrac{1}{3}y(3-y^2)+C.$$
Since $3-y^2 = 2+x^{-2} = x^{-2}(2x^2+1)$, it follows that
$$\int\frac{dx}{x^4\sqrt{x^2-1}} = \frac{(2x^2+1)\sqrt{x^2-1}}{3x^3} + C.$$
e. If $t=\sqrt{2x-9}$, then $x = \tfrac{1}{2}(t^2+9)$ and $dx = t\,dt$. Therefore,
$$\int\frac{dx}{x\sqrt{2x-9}} = \int\frac{2}{(t^2+9)t}\cdot t\,dt = \tfrac{2}{3}\arctan\left(\tfrac{1}{3}\sqrt{2x-9}\right)+C.$$
f. If $t=x^2$, then $\tfrac{1}{2}\,dt = x\,dx$ and $x^4=t^2$, and repeated partial integration gives
$$\int x^5e^{-x^2}\,dx = \tfrac{1}{2}\int t^2e^{-t}\,dt = \tfrac{1}{2}\left(-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right)+C$$
$$= -\tfrac{1}{2}e^{-x^2}(x^4+2x^2+2)+C.$$

2. a. One application of l'Hôpital's Rule gives
$$\lim_{x\to\infty}\frac{\tan^{-1}(2x)-\tfrac{1}{2}\pi}{\sin(1/x)} = \lim_{x\to\infty}\frac{-2}{(x^{-2}+4)\cos(1/x)} = -\tfrac{1}{2}.$$
b. Since
$$\lim_{x\to0}\left(1+\sin^2(x)\right)^{1/\sin^2(x)} = e, \qquad \text{and} \qquad \lim_{x\to0}\left(\frac{\sin x}{x}\right)^2 = 1,$$
it follows that
$$\lim_{x\to0}\left(1+\sin^2(x)\right)^{4/x^2} = e^4.$$

3. a. Integrating by inspection gives
$$\int_{-1}^{0}\frac{e^{1/x}}{x^2}\,dx = \lim_{\delta\to0^-}\int_{-1}^{\delta}\frac{e^{1/x}}{x^2}\,dx = \lim_{\delta\to0^-}\left(-e^{1/x}\right)\Big|_{-1}^{\delta} = e^{-1},$$
since $1/\delta\to-\infty$, and hence $e^{1/\delta}\to0$, as $\delta\to0^-$.
b. If $y = x\log x - x$, then $dy = \log x\,dx$, $y=e^2$ if $x=e^2$ and $y\to\infty$ as $x\to\infty$. Therefore,
$$\int_{e^2}^{\infty}\frac{\log x}{(x\log x - x)^2}\,dx = \lim_{\alpha\to\infty}\int_{e^2}^{\alpha}\frac{dy}{y^2} = \lim_{\alpha\to\infty}\left(-\frac{1}{y}\right)\Big|_{e^2}^{\alpha} = e^{-2}.$$

4. If $y = \tfrac{1}{2}x^2 + \tfrac{1}{4}\log(x)$, then
$$\frac{dy}{dx} = x + \tfrac{1}{4}x^{-1} = \frac{4x^2+1}{4x},$$
and hence
$$1+\left(\frac{dy}{dx}\right)^2 = \frac{16x^4+8x^2+1+16x^2}{16x^2} = \frac{(4x^2+3)^2-8}{16x^2}.$$
If $y=4x^2+3$, then $\tfrac{1}{8}\,dy = x\,dx$, $y=7$ if $x=1$ and $y=19$ if $x=2$, so the length of the curve is
$$\int_1^2\frac{\sqrt{(4x^2+3)^2-8}}{4x}\,dx = \tfrac{1}{8}\int_7^{19}\frac{\sqrt{(y^2-8)}}{y-3}\,dy = \tfrac{1}{8}\int_7^{19}\frac{y^2-8}{(y-3)\sqrt{(y^2-8)}}\,dy$$
$$= \tfrac{1}{8}\int_7^{19}\left\{\frac{y+3}{\sqrt{(y^2-8)}} + \frac{1}{(y-3)\sqrt{(y^2-8)}}\right\}dy,$$
since $y^2-8 = (y-3)(y+3)+1$. Integrating by inspection gives
$$\int_7^{19}\frac{y+3}{\sqrt{(y^2-8)}}\,dy = \left\{\sqrt{(y^2-8)} + 3\log(y+\sqrt{(y^2-8)})\right\}\Big|_7^{19}$$
$$= \sqrt{353}-\sqrt{41}+3\log\left(\frac{19+\sqrt{353}}{7+\sqrt{41}}\right).$$
If $(z-3)^{-1} = y-3$, then $dy = -(z-3)^{-2}\,dz$ and
$$y^2-8 = \frac{1}{(z-3)^2} + \frac{6}{z-3} + 1 = \frac{z^2-8}{(z-3)^2},$$
and hence
$$\int_7^{19}\frac{dy}{(y-3)\sqrt{(y^2-8)}} = \int_{49/16}^{13/4}\frac{dz}{\sqrt{(z^2-8)}} = \log(z+\sqrt{(z^2-8)})\Big|_{49/16}^{13/4}$$
$$= \log\left(\frac{4(13+\sqrt{41})}{49+\sqrt{353}}\right).$$
Therefore, the length of the curve defined by $y = \tfrac{1}{2}x^2 + \tfrac{1}{4}\log(x)$, for $1\leqslant x\leqslant 2$, is equal to
$$\tfrac{1}{8}(\sqrt{353}-\sqrt{41}) + \tfrac{3}{8}\log\left(\frac{19+\sqrt{353}}{7+\sqrt{41}}\right) + \tfrac{1}{8}\log\left(\frac{4(13+\sqrt{41})}{49+\sqrt{353}}\right).$$

5. The curves meet where $0 = x^2 - 2\sqrt{2x} = (x\sqrt{x}-2\sqrt{2})\sqrt{x}$, *i.e.*, at the origin and at the point $(2,2)$. If $0<x<2$, then $x\sqrt{x} < 2\sqrt{2}$, and hence $\tfrac{1}{2}x^2 < \sqrt{2x}$.
a. The area of $\mathscr{R}$ is equal to
$$\int_0^2\left(\sqrt{2x}-\tfrac{1}{2}x^2\right)dx = \left(\tfrac{1}{3}(2x)^{3/2} - \tfrac{1}{6}x^3\right)\Big|_0^2 = \tfrac{4}{3}.$$

b. i. The solid obtained by revolving $\mathscr{R}$ about the x-axis consists of annuli of inner radius $\frac{1}{2}x^2$ and outer radius $\sqrt{2x}$, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$\pi \int_0^2 \left(2x - \tfrac{1}{4}x^4\right) dx.$$

ii. If $\mathscr{R}$ is revolved about the the line defined by $x = -2$, the resulting solid consists of cylindrical shells of radius $x+2$ and height $\sqrt{2x} - \frac{1}{2}x^2$, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$2\pi \int_0^2 (x+2)\left(\sqrt{2x} - \tfrac{1}{2}x^2\right) dx.$$

iii. The solid obtained by revolving $\mathscr{R}$ about the line defined by $y = 3$ can be decomposed into annuli of inner radius $3 - \sqrt{2x}$ and outer radius $3 - \frac{1}{2}x^2$, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$\pi \int_0^2 \left\{\left(3 - \tfrac{1}{2}x^2\right)^2 - \left(3 - \sqrt{2x}\right)^2\right\} dx.$$

6. The given equation is equivalent to

$$x(y^2+1) + y(x^2-1)\frac{dy}{dx} = 0, \qquad \text{or} \qquad \frac{d}{dx}\Big((x^2-1)(y^2+1)\Big) = 0,$$

so there is a real number a such that $(x^2-1)(y^2+1) = a$, at least if $-1 < x < 1$. The condition $y = -2$ if $x = 0$ implies that $a = -5$, and so $(1-x^2)(1+y^2) = 5$. If it is required to solve for y, then dividing by $1 - x^2$ gives

$$1 + y^2 = \frac{5}{1-x^2}, \quad \text{or} \quad y^2 = \frac{x^2+4}{1-x^2}, \quad \text{and so} \quad y = -\sqrt{\frac{x^2+4}{1-x^2}}.$$

7. If there are y kilograms of salt in the tank after t minutes, then

$$\frac{dy}{dt} = -\frac{y}{100+5t}\cdot 10, \qquad \text{or} \qquad \frac{1}{y}\frac{dy}{dt} = -\frac{2}{t+20}.$$

Hence, there is a real number $a = e^c$ such that

$$\log y = -2\log(t+20) + c, \qquad \text{or} \qquad y = \frac{a}{(t+20)^2},$$

and $a = 3600$, since $y = 9$ when $t = 0$. Therefore, after ten minutes there are $3600(10+20)^{-2} = 4$ kilograms of salt in the tank.

8. a. If $a_n = 2^n/n!$, then the given terms are $a_0, a_1, a_2, a_3, a_4, a_5, a_6$, and $\lim a_n = 0$ since if $n \geqslant 1$ then

$$0 < a_n = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdots\frac{2}{n} \leqslant \frac{4}{n}.$$

b. If $a_n = 2^n/n! + n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$, then the given terms are $a_0, a_1, a_2, a_3, a_4, a_5, a_6$, and $\lim a_n = \infty$.

c. If $a_n = 2^{n \pmod 7}/(n! \pmod 7)$, where $k \pmod 7$ is remainder when the integer k is divided by 7, then the given terms are $a_0, a_1, a_2, a_3, a_4, a_5, a_6$. Since $a_{n+7} = a_n$ for $n \geqslant 0$, the sequence $\{a_n\}$ oscillates and diverges.

Note: Since the first seven terms of a sequence give no information about its long run behaviour, there are many possible answers to this question.

9. a. Since $0 < n^{-1} < n^{-1}e^{3/n^2}$ if $n \geqslant 1$, the comparison test implies that the series $\sum n^{-1}e^{3/n^2}$ diverges with the harmonic series.

b. If $k \geqslant 2$,

$$a_k = \frac{1}{3k\sqrt{\log k} + 5k\sqrt[3]{(\log k)^2} - 7\log k} \qquad \text{and} \qquad b_k = \frac{1}{k\sqrt[3]{(\log k)^2}},$$

then $a_k, b_k > 0$ and $\sum b_k$ is a divergent logarithmic p-series ($p = \frac{2}{3} \leqslant 1$). Since

$$\lim \frac{a_k}{b_k} = \lim \frac{1}{3(\log k)^{-1/6} + 5 - 7k^{-1}(\log k)^{1/3}} = \frac{1}{5}$$

(from elementary properties of the logarithm), the limit comparison test implies that $\sum a_k$ is divergent.

c. If $n \geqslant 1$ then

$$0 < \left(\frac{4n-1}{25n+1}\right)^{n/2} < \left(\frac{4}{25}\right)^{n/2} = \left(\frac{2}{5}\right)^n,$$

so the comparison test implies that the given series converges with the geometric series $\sum\left(\frac{2}{5}\right)^n$ (whose ratio, $\frac{2}{5}$, is positive and smaller than 1).

d. Since

$$\lim \frac{n\cos(3/n)}{2n+1} = \frac{1}{2} \neq 0, \qquad \text{the series} \qquad \sum_{n=1}^{\infty} \frac{n\cos(3/n)}{2n+1}$$

is divergent by the vanishing condition.

10. a. If

$$a_n = \left(\frac{2}{e}\right)^n \qquad \text{and} \qquad b_n = \frac{(-2)^n}{e^n + 1 - n^2} = \left(-\frac{2}{e}\right)^n \cdot \frac{1}{1 + (1-n^2)e^{-n}}$$

then $\lim\{a_n \cdot |b_n|^{-1}\} = 1$, since $\lim\{(1-n^2)e^{-n}\} = 0$ by elementary properties of the exponential function. Since $\sum a_n$ is a convergent geometric series (its ratio is $2/e$, and $0 < 2/e < 1$), the limit comparison test implies that the series $\sum b_n$ is absolutely convergent.

b. If $n \geqslant 1$, then

$$c_n = \frac{1}{\sqrt{n + 2\sqrt{n+1}}} > \frac{1}{3\sqrt{n+1}} = \frac{1}{3\sqrt{1+1/n}}\cdot\frac{1}{\sqrt{n}} \geqslant \frac{1}{6}\cdot\frac{1}{\sqrt{n}},$$

so the comparison test implies that the series $\sum c_n$ diverges with the p-series $\sum n^{-1/2}$ ($p = \frac{1}{2} \leqslant 1$). However, $\lim c_n = 0$ and if $n \geqslant 1$, then

$$c_n = \frac{1}{\sqrt{n + 2\sqrt{n+1}}} > \frac{1}{\sqrt{n+1+2\sqrt{n+2}}} = c_{n+1},$$

so the alternating series test implies that the series $\sum(-1)^n c_n$ is convergent. Therefore, $\sum(-1)^n c_n$ is conditionally convergent.

c. If $a_k = k^{-1/4}(\log k)^2$, then $a_k > k^{-1/4}$ if $k \geqslant 3$, so the series $\sum a_k$ diverges with the p-series $\sum k^{-1/4}$ ($p = \frac{1}{4} \leqslant 1$) by the comparison test. However, $\lim a_k = 0$ (by elementary properties of the logarithm), and if $k > e^8$ then

$$\frac{d}{dk}\left\{\frac{(\log k)^2}{\sqrt[4]{k}}\right\} = \frac{2\log k}{k\sqrt[4]{k}} - \frac{(\log k)^2}{4k\sqrt[4]{k}} = \frac{(\log k)(8 - \log k)}{4k\sqrt[4]{k}} < 0,$$

and hence $a_{k+1} < a_k$. So the alternating series test implies that $\sum(-1)^k a_k$ is convergent. Therefore, the series $\sum(-1)^k a_k$ is conditionally convergent.

d. If $n \geqslant 1$ then

$$0 < d_n = \frac{(2n)!}{(2n)^{2n}} = \frac{2n}{2n}\cdot\frac{2n-1}{2n}\cdots\frac{n}{2n}\cdot\frac{n-1}{2n}\cdots\frac{3}{2n}\cdot\frac{2}{2n}\cdot\frac{1}{2n} \leqslant \frac{1}{2^n},$$

so the comparison test implies that $\sum d_n$ converges with the geometric series $\sum 2^{-n}$ (whose ratio is $\frac{1}{2}$). Therefore, $\sum(-1)^n d_n$ is absolutely convergent.

Note. — The ratio test could also be used, in which case $\lim \frac{d_{n+1}}{d_n} = e^{-2}$.

11. Let

$$\alpha_n = (-1)^n \frac{\sin(1/\sqrt{n})}{5^n\sqrt{n+1}}(x+2)^n.$$

If $x \neq -2$, then

$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \frac{1}{5}\lim\left\{\frac{\sin(1/\sqrt{(n+1)})}{1/\sqrt{(n+1)}}\cdot\frac{1/\sqrt{n}}{\sin(1/\sqrt{n})}\cdot\frac{\sqrt{n}}{\sqrt{n+2}}\cdot|x+2|\right\} = \frac{1}{5}|x+2|,$$

so the ratio test implies that $\sum\alpha_n$ is absolutely convergent if $\frac{1}{5}|x+2| < 1$, *i.e.*, $-7 < x < 3$, and diverges if $x < -7$ or $x > 3$. Let

$$a_n = \frac{\sin(1/\sqrt{n})}{\sqrt{n+1}}, \qquad \text{and} \qquad b_n = \frac{1}{n}.$$

If $x = -7$, then $\alpha_n = a_n$, and

$$\lim \frac{a_n}{b_n} = \lim\left\{\frac{\sin(1/\sqrt{n})}{1/\sqrt{n}}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\right\} = 1,$$

so $\sum a_n$ diverges with the harmonic series by the limit comparison test. Hence, -7 does not belong to the interval of convergence of $\sum \alpha_n$. If $x = 3$, then $\alpha_n = (-1)^n a_n$, $\lim a_n = 0$, and if $n \geqslant 1$, then

$$a_n = \frac{\sin(1/\sqrt{n})}{\sqrt{n+1}} > \frac{\sin(1/\sqrt{(n+1)})}{\sqrt{n+2}} = a_{n+1}$$

because the numerator is diminished while the denominator grows. So the alternating series test implies that $\sum(-1)^n a_n$ is convergent; thus, 3 belongs to the interval of convergence of $\sum \alpha_n$. Therefore, the power series $\sum \alpha_n$ has radius of convergence 5 and interval of convergence $(-7, 3]$.

12. Differentiating a geometric series yields

$$\begin{aligned}\frac{1}{(1-2x)^2} &= \tfrac{1}{2}\frac{d}{dx}\left\{\frac{1}{1-2x}\right\} = \tfrac{1}{2}\frac{d}{dx}\sum_{k=0}^{\infty}(2x)^k = \sum_{k=0}^{\infty}(k+1)2^k x^k \\ &= 1 + 4x + 12x^2 + 32x^3 + 80x^4 + \cdots,\end{aligned}$$

which is valid if $|2x| < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$.

13. a. By a direct calculation,

$$\begin{aligned} f(x) &= \frac{d}{dx}\left\{x\arccos(5x) - \tfrac{1}{5}\sqrt{1-25x^2}\right\} \\ &= \arccos(5x) - \frac{5x}{\sqrt{1-25x^2}} - \frac{1}{5}\cdot\frac{-25x}{\sqrt{1-25x^2}} \\ &= \arccos(5x).\end{aligned}$$

b. If $0 \leqslant \vartheta \leqslant \frac{1}{2}\pi$ and $x = \sin^2(\vartheta)$, then $dx = 2\sin(\vartheta)\cos(\vartheta)\,d\vartheta = \sin(2\vartheta)\,d\vartheta$ and $0 \leqslant x \leqslant 1$. Therefore,

$$\int_0^{\frac{1}{2}\pi} e^{\sin\vartheta}\sin(2\vartheta)\,d\vartheta = \int_0^1 e^{\sqrt{x}}\,dx.$$

Since the integrands are non-negative, this shows the equality of the areas in question.

c. i. If $\sum a_n$ converges, then $\lim a_n = 0$ by the vanishing condition, and thus $\{(a_n)^{-1}\}$ does not converge to zero (because $(a_n)^{-1}a_n = 1$ if $a_n \neq 0$). Hence, the vanishing condition implies that $\sum(a_n)^{-1}$ is divergent.

ii. If $\lim(n^2 a_n) = 0$ then there is a positive integer k such that $|n^2 a_n| < 1$, or $|a_n| < n^{-2}$, if $n \geqslant k$. So the comparison test implies that the series $\sum a_n$ converges with the p-series $\sum n^{-2}$ $(p = 2 > 1)$.

iii. If $\sum|a_n|$ converges then $\lim a_n = 0$, so there is a positive integer k such that $|a_n| < \frac{1}{2}$, thus $\frac{1}{2} < 1 - a_n < \frac{3}{2}$, and hence $0 \leqslant |a_n\cos(n)(1-a_n)^{-1}| < 2|a_n|$, if $n \geqslant k$. So the comparison test implies that the series $\sum a_n\cos(n)(1-a_n)^{-1}$ is (absolutely) convergent.