1. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{\cos^3(x)}{\sqrt{\sin(x)}}\,dx$ b. $\displaystyle\int \frac{x\arcsin(x^2)}{\sqrt{1-x^4}}\,dx$ c. $\displaystyle\int \frac{x+6}{x(x^2+2x+3)}\,dx$

d. $\displaystyle\int \sin(\log x)\,dx$ e. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-4}}$ f. $\displaystyle\int \sqrt{\frac{3+x}{3-x}}\,dx$

2. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 0^+}\frac{\log(\sin x)}{\log(\sin 2x)}$ b. $\displaystyle\lim_{x\to\frac{1}{2}\pi^-}(\tan x)^{2x-\pi}$ c. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{3}{e^{3x}-1}\right)$

3. Determine whether each improper integral is convergent or divergent. If an integral is convergent, evaluate it.

a. $\displaystyle\int_0^\infty (-xe^{-x})\,dx$ b. $\displaystyle\int_0^2 \frac{dx}{(x-1)^{2/3}}$

4. Give the solution of the differential equation

$$\cos(x)\,\frac{dy}{dx} = \sin(x)\sqrt{y^2+4}$$

which satisfies $y = 0$ if $x = 0$.

5. Find the area of the region enclosed by the curves defined by

$$y = x^3 + x^2 + 3x + 1 \qquad \text{and} \qquad y = x^3 + x + 4.$$

6. Let $\mathscr{R}$ be the region enclosed by the graphs of $x = 0$, $y = x + 1$ and $y = x^3 + x$. Write an integral which gives the volume of the solid obtained by revolving $\mathscr{R}$ about: a. the x axis; b. the line defined by $x = 3$.

7. Find the arc length function of the curve $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y$, taking $P\left(\frac{1}{4}, 1\right)$ as the starting point.

8. Determine whether each sequence converges or diverges. If a sequence converges find its limit; otherwise, explain why it diverges.

a. $\displaystyle\left\{\left(\frac{3n+1}{3n-1}\right)^n\right\}_{n=1}^{\infty}$ b. $\displaystyle\left\{\frac{n^3(2n)!}{(2n+2)!}\right\}_{n=0}^{\infty}$

9. For the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right),$$

a. give a formula for the sum of the first n terms of the series, and

b. find the sum of the series.

10. Determine whether each series is convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=0}^{\infty}\frac{\sqrt{n^2+3}}{3n^2+7}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{\log(n)}{n\sqrt{n}}$

11. Determine whether each series is absolutely or conditionally convergent, or divergent. Justify your conclusions.

a. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n!}{1\cdot 3\cdot 5\cdots(2n+1)}$ b. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^k}{3^{k+1}}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{\sqrt{5n+3}}$

12. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x+1)^n}{5^n\sqrt{n}}.$$

13. Find the Taylor series of

$$f(x) = \frac{1}{2+x}$$

centred at 1. Write the first four terms of the series explicitly, and express the series using appropriate sigma notation.

Solutions

1. a. If $y = \sqrt{\sin(x)}$ then $2y\,dy = \cos(x)\,dx$ and $\cos^2(x) = 1 - y^4$, so

$$\int \frac{\cos^3(x)}{\sqrt{\sin(x)}}\,dx = 2\int (1-y^4)\,dy = 2\left(y - \tfrac{1}{5}y^5\right) + a$$
$$= \tfrac{2}{5}(5 - \sin^2(x))\sqrt{\sin(x)} + \alpha.$$

b. If $y = \arcsin(x^2)$ then $dy = 2x(1-x^4)^{-1/2}\,dx$, so

$$\int \frac{x\arcsin(x^2)}{\sqrt{1-x^4}}\,dx = \tfrac{1}{2}\int y\,dy = \tfrac{1}{4}\left(\arcsin(x^2)\right)^2 + \beta.$$

c. The resolution into partial fractions of the integrand is

$$\frac{x+6}{x(x^2+2x+3)} = \frac{2}{x} - \frac{2x+3}{x^2+2x+3},$$

where the first coefficient is found by inspection (covering and evaluating) and the second an third coefficients are obtained by comparing the quadratic and linear terms of the numerator. The integral of the second partial fraction is

$$\int \frac{(2x+2)+1}{x^2+2x+3}\,dx = \log(x^2+2x+3) + \tfrac{1}{2}\sqrt{2}\arctan\left(\tfrac{1}{2}\sqrt{2}(x+1)\right) + c,$$

and therefore

$$\int \frac{x+6}{x(x^2+2x+3)}\,dx = \log\frac{x^2}{x^2+2x+3} - \tfrac{1}{2}\sqrt{2}\arctan\left(\tfrac{1}{2}\sqrt{2}(x+1)\right) + \gamma.$$

d. Repeated partial integration and absorption gives

$$\int \sin(\log x)\,dx = x\sin(\log x) - x\cos(\log x) - \int \sin(\log x)\,dx$$
$$= \tfrac{1}{2}x(\sin(\log x) - \cos(\log x)) + \delta.$$

e. If $y = \sqrt{x^2-4}$, then $y^2 = x^2 - 4$, so $y\,dy = x\,dx$, or $dx/(xy) = dy/(x^2)$, and thus

$$d\left(\frac{y}{x^2}\right) = \frac{dy}{x^2} - \frac{2y\,dx}{x^3} = \frac{dy}{x^2} - \frac{2(x^2-4)}{x^3y}\,dx = \frac{8\,dx}{x^3y} - \frac{dy}{y^2+4}.$$

Therefore,

$$\int \frac{dx}{x^3\sqrt{x^2-4}} = \frac{y}{8x^2} + \frac{1}{8}\int \frac{dy}{y^2+4} = \frac{\sqrt{x^2-4}}{8x^2} + \frac{1}{16}\arctan\left(\tfrac{1}{2}\sqrt{x^2-4}\right) + \varepsilon.$$

f. Multiplying and dividing by $\sqrt{3+x}$ omits -3 from the domain of the integrand, and gives

$$\int \frac{3+x}{\sqrt{9-x^2}}\,dx = 3\arcsin\left(\tfrac{1}{3}x\right) - \int dy = 3\arcsin\left(\tfrac{1}{3}x\right) - \sqrt{9-x^2} + \zeta,$$

where in the second term $y = \sqrt{9-x^2}$, so that $dy = -(x/y)\,dx$.

2. a. Since $\lim_{\vartheta\to 0}\frac{\sin\vartheta}{\vartheta} = 1$, revising the expression in the limit gives

$$\lim_{x\to 0^+}\frac{\log(\sin x)}{\log(\sin 2x)} = \lim_{x\to 0^+}\frac{1 + \log\left(\frac{\sin x}{x}\right)/(\log x)}{1 + \log\left(2\frac{\sin 2x}{2x}\right)/(\log x)} = 1.$$

b. If $y = \frac{1}{2}\pi - x$ then $(\tan x)^{2x-\pi} = (\tan y)^{2y}$, so multiplying and dividing by $y^y = e^{y\log y}$ gives

$$\lim_{x\to\frac{1}{2}\pi^-}(\tan x)^{2x-\pi} = \lim_{y\to 0^+}(\tan y)^{2y} = \lim_{y\to 0^+}\left\{\left(\frac{\sin y}{y\cos y}\right)^y e^{y\log y}\right\}^2 = 1,$$

by elementary properties of the logarithm (the definition of $\log y$ implies that $-1/y < 1 - 1/y < \log y < 0$ provided $0 < y < 1$, which immediately gives $0 < y^b(-\log y)^a < (2a/b)^a y^{b/2}$ for $0 < y < 1$ and $a, b > 0$) and $\lim_{\vartheta\to 0}\frac{\sin\vartheta}{\vartheta} = 1$.

c. Combining terms and expanding the exponential function gives

$$\lim_{x\to 0}\frac{e^{3x} - 1 - 3x}{x(e^{3x}-1)} = \lim_{x\to 0}\frac{\frac{9}{2} + \frac{9}{2}x + \frac{27}{8}x^2 + \cdots}{3 + \frac{9}{2}x + \frac{9}{2}x^2 + \cdots} = \frac{3}{2}.$$

(Alternatively, two applications of l'Hôpital's rule could be used.)

3. a. Partial integration gives

$$\int_0^\infty (-xe^{-x})\,dx = \lim_{t\to\infty}(x+1)e^{-x}\Big|_0^\infty = \lim_{t\to\infty}\left\{\frac{t+1}{e^t} - 1\right\} = -1,$$

by basic properties of the exponential function (the inequality proved in Part c of Question 2 gives $0 < y^a e^{-by^c} < (2a/(bc))^{a/c} e^{-\frac{1}{2}by^c}$ for $a, b, c, y > 0$).

b. Integrating by inspection gives

$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = \lim_{s\to 1^-} 3\sqrt[3]{(x-1)}\Big|_0^s + \lim_{t\to 1^+} 3\sqrt[3]{(x-1)}\Big|_t^2 = 6.$$

4. For $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, the equation in question is equivalent to

$$\frac{1}{\sqrt{(y^2+4)}}\frac{dy}{dx} = \tan x, \qquad \text{or} \qquad \log(y + \sqrt{(y^2+4)}) = \log(\sec x) + C,$$

which is equivalent to $y + \sqrt{(y^2+4)} = A\sec(x)$ (where $A = e^C$). If $y = 0$ when $x = 0$ then $A = 2$, so $y + \sqrt{(y^2+4)} = 2\sec(x)$. Subtracting y and squaring then gives $4\sec^2(x) - 4y\sec(x) = 4$, or $y = \sec(x) - \cos(x) = \sin(x)\tan(x)$.

5. If $\overline{y} = x^3 + x + 4$ and $\underline{y} = x^3 + x^2 + 3x + 1$ then $\overline{y} - \underline{y} = (3+x)(1-x)$, which is positive if $-3 < x < 1$ and vanishes if x is -3 or 1. So the area of the region enclosed by the curves is

$$\int_{-3}^{1}(\overline{y} - \underline{y})\,dx = \int_{-3}^{1}(-x^2 - 2x + 3)\,dx = \left(-\tfrac{1}{3}x^3 - x^2 + 3x\right)\Big|_{-3}^{1}$$
$$= -\tfrac{28}{3} + 8 + 12 = \tfrac{32}{3}.$$

6. If $\overline{y} = x + 1$ and $\underline{y} = x^3 + x$ then $\overline{y} - \underline{y} = 1 - x^3$, which is positive if $0 < x < 1$.

a. The solid obtained by revolving $\mathscr{R}$ about the x axis consists of annuli of inner radius $x^3 + x$ and outer radius $x + 1$, for $0 \leqslant x \leqslant 1$, so its volume is

$$\pi\int_0^1 \left\{(x+1)^2 - (x^3+x)^2\right\}dx.$$

b. The solid obtained by rotating $\mathscr{R}$ about the line defined by $x = 3$ consists of cylindrical shells of radius $3 - x$ and height $1 - x^3$, for $0 \leqslant x \leqslant 1$, so its volume is

$$2\pi\int_0^1 (3-x)(1-x^3)\,dx.$$

7. If $x = \frac{1}{4}y^2 - \frac{1}{2}\log y$, then

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\tfrac{1}{2}y - \tfrac{1}{2}y^{-1}\right)^2 = \tfrac{1}{4}y^2 + \tfrac{1}{2} + \tfrac{1}{4}y^{-2} = \left(\tfrac{1}{2}y + \tfrac{1}{2}y^{-1}\right)^2,$$

and hence

$$\int_1^y \sqrt{1 + \left(\frac{dx}{d\eta}\right)^2}\,d\eta = \tfrac{1}{2}\int_1^y (\eta + \eta^{-1})\,d\eta = \tfrac{1}{4}(y^2 - 1) + \tfrac{1}{2}\log y,$$

which is the length of the curve between $\left(\frac{1}{4}, 1\right)$ and (x, y) if $y \geqslant 1$, and is -1 times the length of the curve between $\left(\frac{1}{4}, 1\right)$ and (x, y) if $0 < y < 1$.

8. a. Since

$$\lim_{t\to 0}(1+t)^{1/t} = e, \qquad \lim_{n\to\infty}\frac{2n}{3n-1} = \frac{2}{3}$$

and

$$a_n = \left(\frac{3n+1}{3n-1}\right)^n = \left(1 + \frac{2}{3n-1}\right)^{\frac{3n-1}{2}\cdot\frac{2n}{3n-1}},$$

it follows that $\lim_{n\to\infty} a_n = e^{2/3}$.

b. Since
$$a_n = \frac{n^3(2n)!}{(2n+2)!} = \frac{n^3}{(2n+2)(2n+1)} = \frac{n}{2(1+1/n)(2+1/n)},$$
it follows that the sequence $\{a_n\}$ diverges to ∞.

9. If
$$a_n = \frac{1}{n} - \frac{1}{n+2} \qquad \text{and} \qquad A_n = \frac{1}{n} + \frac{1}{n+1},$$
then $a_n = A_n - A_{n+1}$ for $n \geqslant 1$, and the sum of the first n terms of the series is
$$a_1 + a_2 + \cdots + a_n = A_1 - A_{n+1} = \tfrac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$
Hence, the sum of the series is $\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = \frac{3}{2}$.

10. a. If $n \geqslant 1$ then
$$a_n = \frac{\sqrt{n^2+3}}{3n^2+7} > \frac{\sqrt{n^2}}{3n^2+7n^2} = \frac{1}{10n} > 0,$$
so the comparison test implies that $\sum a_n$ diverges with the harmonic series.

b. As $\frac{d}{dx}(x^{-1/4}\log x) = \frac{1}{4}x^{-5/4}(4-\log x)$ is positive if $0 < x < e^4$ and negative if $x > e^4$, it follows that
$$0 \leqslant a_n = \frac{\log n}{n^{3/2}} = \frac{\log n}{n^{1/4}} \cdot \frac{1}{n^{5/4}} < \frac{4}{e} \cdot \frac{1}{n^{5/4}},$$
for $n \geqslant 1$. Therefore, the comparison test implies that $\sum a_n$ converges with the p-series $\sum n^{-5/4}$.

11. a. Since $\sum 2^{-n}$ is a convergent geometric series, and
$$0 < a_n = \frac{n!}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)} = 1\cdot\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\cdots\frac{n}{2n+1} < \left(\frac{1}{2}\right)^n,$$
for $n \geqslant 1$, the comparison test implies that the series $\sum(-1)^n a_n$ is absolutely convergent.

b. If $n > 3$ then
$$a_n = \frac{n^n}{3^{n+1}} = \frac{1}{3}\cdot\frac{n}{3}\cdot\frac{n}{3}\cdot\frac{n}{3}\cdots\frac{n}{3} > \frac{n}{9} > 0, \qquad \text{so} \qquad \lim_{n\to\infty} a_n = \infty.$$
Hence, the vanishing condition implies that $\sum(-1)^n a_n$ is divergent.

c. If $n \geqslant 1$ then
$$a_n = \frac{1}{\sqrt{5n+3}} \geqslant \frac{1}{\sqrt{5n+3n}} = \frac{\sqrt{2}}{4}\cdot\frac{1}{\sqrt{n}} > 0,$$
so the comparison test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$. On the other hand,
$$a_n = \frac{1}{\sqrt{5n+3}} > \frac{1}{\sqrt{5n+8}} = a_{n+1}$$
if $n \geqslant 1$, and $\lim a_n = 0$, so the alternating series test implies that $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum\cos(n\pi)a_n = \sum(-1)^n a_n$ is conditionally convergent.

12. If $x \neq -1$ and
$$\alpha_n = \frac{(-1)^n(x+1)^n}{5^n\sqrt{n}},$$
then
$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim\frac{1}{5\sqrt{1+1/n}}|x+1| = \tfrac{1}{5}|x+1|,$$
so the ratio test implies that $\sum\alpha_n$ is absolutely convergent if $|x+1| < 5$, *i.e.*, $-6 < x < 4$, and divergent if $x < -6$ or $x > 4$. If $x = -6$ then $\sum\alpha_n = \sum n^{-1/2}$ is a divergent p-series ($p = \frac{1}{2} \leqslant 1$), and if $x = 4$ then $\sum\alpha_n = \sum(-1)^n n^{-1/2}$, which is convergent by the alternating series test ($n^{-1/2} > (n+1)^{-1/2}$ if $n \geqslant 1$, and $\lim n^{-1/2} = 0$). Therefore, the radius of convergence of $\sum\alpha_n$ is 5, and the interval of convergence of $\sum\alpha_n$ is $(-6, 4]$.

13. From the expansion $1/(1+t) = \sum_{k=0}^{\infty}(-1)^k t^k$ (a geometric series), it follows that
$$\frac{1}{2+x} = \frac{1}{3}\cdot\frac{1}{1+\frac{1}{3}(x-1)} = \frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{3^k}(x-1)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{3^{k+1}}(x-1)^k$$
$$= \tfrac{1}{3} - \tfrac{1}{9}(x-1) + \tfrac{1}{27}(x-1)^2 - \tfrac{1}{81}(x-1)^3 + \cdots,$$
which is valid if $\frac{1}{3}|x-1| < 1$, or equivalently, $-2 < x < 4$.