Question 1. — Evaluate each of the following integrals.

a. $\displaystyle\int \frac{2x^2+x+3}{x(x+1)^2}\,dx$ b. $\displaystyle\int \frac{\cot^3(\log x)\csc^4(\log x)}{x}\,dx$ c. $\displaystyle\int \frac{x^2}{\sqrt{25-x^2}}\,dx$

d. $\displaystyle\int \frac{2x+5}{4x^2+4x+10}\,dx$ e. $\displaystyle\int_1^6 e^{\sqrt{3x-2}}\,dx$ f. $\displaystyle\int e^x \arcsin(e^x)\,dx$

Question 2. — Evaluate the following limits.

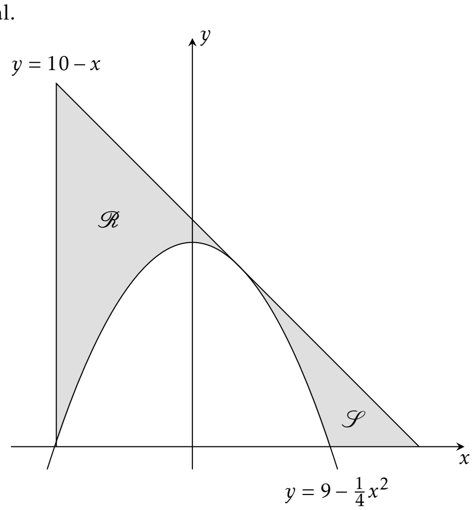
a. $\displaystyle\lim_{x\to 0^+} \frac{\arctan(\sqrt{x})}{3\sqrt{x}}$ b. $\displaystyle\lim_{x\to 0^+} (\sin x)^x$

Question 3. — Evaluate each improper integral or show that it diverges.

a. $\displaystyle\int_0^\infty \frac{e^{-2x}}{e^{-2x}+3}\,dx$ b. $\displaystyle\int_0^{\frac{1}{3}\pi} \tan(2x)\,dx$

Question 4. — Find the value of $k>0$ such that the region enclosed by the graphs of $y=x^2-kx$ and $y=\frac{1}{2}x^2$ has area 18.

Question 5. — Let $\mathscr{R}$ and $\mathscr{S}$ be the regions labelled in the figure below, in which the slanted line is tangent to the parabola and the line to the left of $\mathscr{R}$ is vertical.



Write an integral which is equal to the volume of the solid obtained by revolving: a. $\mathscr{R}$ about the x-axis; b. $\mathscr{S}$ about the line defined by $y=-1$.

Question 6. — Solve the initial value problem

$$\frac{dy}{dx} = xy\sqrt{1+x^2}, \qquad \text{given} \quad y(0)=-1.$$

Question 7. — There are initially 100 bacteria in a petri dish. After two hours, the number of bacteria has tripled. Assuming that the population of bacteria in the dish increases at a rate proportional to its size, determine the population of bacteria in the dish after t hours.

Question 8. — Find the length of the curve defined by

$$x = \tfrac{1}{3}y^{3/2} - y^{1/2}, \qquad 1 \leqslant y \leqslant 4.$$

Question 9. — Determine whether the sequence converges or diverges. If a sequence converges, compute its limit.

a. $\displaystyle\left\{\arcsin\left(\frac{3n-2}{5-6n}\right)\right\}$ b. $\displaystyle\left\{\int_{n+1}^{n+4} \frac{dx}{x+3}\right\}$

Question 10. — Suppose that $\{a_n\}$ is a sequence of positive numbers, and that the series $\sum a_n$ is convergent. Determine whether the following are convergent or divergent. Justify each answer completely.

a. $\{a_n\}$ b. $\displaystyle\sum_{n=1}^\infty \frac{a_n-1}{3+a_n}$ c. $\displaystyle\sum_{n=1}^\infty (-1)^n \arctan(a_n)$

Question 11. — Find the sum of the series $\displaystyle\sum_{n=0}^\infty \Big\{\arctan(n+1)-\arctan(n)\Big\}$.

Question 12. — Determine whether each series is convergent or divergent. Justify your answers completely.

a. $\displaystyle\sum_{n=1}^\infty \frac{e^n+3^n}{5^n-2^n}$ b. $\displaystyle\sum_{n=1}^\infty \frac{\sin(1/n)}{n}$

Question 13. — Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your answers completely.

a. $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n(2n+1)}{n^2+3n}$ b. $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n+1}(3n-3)!}{3n\cdot 3^{3n}}$ c. $\displaystyle\sum_{n=1}^\infty \left(\frac{5-n}{2n}\right)^n$

Question 14. — Find the interval of convergence of the power series

$$\sum_{n=1}^\infty \frac{3\cdot 5\cdot 7\cdots(2n+1)}{3^n(n+1)!}(x+3)^n.$$

Question 15. — Determine the Taylor series of $\log(x+3)$, centred at 2.

Solution to question 1. — a. The resolution into partial fractions of the integrand is

$$\frac{2x^2+x+3}{x(x+1)^2}=\frac{3}{x}-\frac{4}{(x+1)^2}-\frac{1}{x+1},$$

where the first two coefficients are obtained by inspection (covering and evaluating) and the third is found by comparing the quadratic coefficients of the numerators. Integrating by inspection then gives

$$\int\frac{2x^2+x+3}{x(x+1)^2}\,dx=\log\left|\frac{x^3}{x+1}\right|+\frac{4}{x+1}+\alpha.$$

b. If $c=\cot(\log x)$ then $dc=-\csc^2(\log x)/x\,dx$ and $\csc^2(\log x)=c^2+1$, so the integral is equal to

$$-\int c^3(c^2+1)\,dc=-\tfrac{1}{6}\cot^6(\log x)-\tfrac{1}{4}\cot^4(\log x)+\beta.$$

c. Integrating by parts, and then revising the remaining integrand, gives

$$\begin{aligned}\int x\cdot\frac{x\,dx}{\sqrt{25-x^2}}&=-x\sqrt{25-x^2}+\int\sqrt{25-x^2}\,dx\\&=-x\sqrt{25-x^2}+\int\frac{25}{\sqrt{25-x^2}}\,dx-\int\frac{x^2}{\sqrt{25-x^2}}\,dx.\end{aligned}$$

Absorbing the last integral on the right side then yields

$$\int\frac{x^2}{\sqrt{25-x^2}}\,dx=-\tfrac{1}{2}x\sqrt{25-x^2}+\tfrac{25}{2}\arcsin\left(\tfrac{1}{5}x\right)+\gamma.$$

d. Since $4x^2+4x+10=2(2x^2+2x+5)=(2x+1)^2+9$ and $2x+5=2x+1+4$, integrating by inspection gives

$$\begin{aligned}\int\frac{2x+5}{4x^2+4x+10}\,dx&=\frac{1}{2}\int\frac{2x+1}{2x^2+2x+5}\,dx+\int\frac{4\,dx}{(2x+1)^2+9}\\&=\tfrac{1}{4}\log(2x^2+2x+5)+\tfrac{2}{3}\arctan\left(\tfrac{1}{3}(2x+1)\right)+\delta.\end{aligned}$$

e. If $y=\sqrt{3x-2}$ then $y^2=3x-2$ and $dx=\frac{2}{3}y\,dy$, so partial integration gives

$$\int_1^6 e^{\sqrt{3x-2}}\,dx=\frac{2}{3}\int_1^4 ye^y\,dy=\tfrac{2}{3}e^y(y-1)\Big|_1^4=2e^4.$$

f. If $y=\arcsin(e^x)$, then $\sin(y)=e^x$, $\cos(y)\,dy=e^x\,dx$ and $\cos(y)=\sqrt{1-e^{2x}}$, so the integral is equal to

$$\int y\cos(y)\,dy=y\sin(y)+\cos(y)+\varphi=e^x\arcsin(e^x)+\sqrt{1-e^{2x}}+\varphi.$$

Solution to question 2. — a. If $y=\arctan(\sqrt{x})$, or $\sin(y)/\cos(y)=\sqrt{x}$, then $y\to 0^+$ as $x\to 0^+$, and thus

$$\lim_{x\to 0^+}\frac{\arctan(\sqrt{x})}{3\sqrt{x}}=\frac{1}{3}\lim_{y\to 0^+}\left\{\frac{y}{\sin(y)}\cdot\cos(y)\right\}=\frac{1}{3},$$

since $\sin(y)/y\to 1$ as $y\to 0$.

b. Multiplying and dividing by $x^x=e^{x\log(x)}$ gives

$$\lim_{x\to 0^+}(\sin x)^x=\lim_{x\to 0^+}\left\{\left(\frac{\sin x}{x}\right)^x\cdot e^{x\log(x)}\right\}=1\cdot e^0=1,$$

since $\sin(x)/x\to 1$ and $x\log(x)\to 0$ as $x\to 0^+$ (by elementary properties of the logarithm).

Solution to question 3. — a. Integrating by inspection gives

$$\int_0^\infty\frac{e^{-2x}}{e^{-2x}+3}\,dx=-\frac{1}{2}\lim_{t\to\infty}\log(e^{-2x}+3)\Big|_0^t=\tfrac{1}{2}\log\tfrac{4}{3}.$$

b. The improper integral diverges, since

$$\int_0^{\frac{1}{4}\pi}\tan(2x)\,dx=-\frac{1}{2}\lim_{t\to\frac{1}{4}\pi^-}\log(\cos 2x)\Big|_0^t=\infty.$$

(and similarly the improper integral of diverges to $-\infty$).

Solution to question 4. — Since $\frac{1}{2}x^2-(x^2-kx)=\frac{1}{2}x(2k-x)$ is positive if $0<x<2k$, and otherwise is negative or zero, the area of the region enclosed by the parabolas is

$$\frac{1}{6}\int_0^{2k}(6kx-3x^2)\,dx=\tfrac{1}{6}(3kx^2-x^3)\Big|_0^{2k}=\tfrac{2}{3}k^3,$$

which is equal to 18 if, and only if, $k^3=27$, or $k=3$.

Solution to question 5. — By inspection, the parabola has x intercepts ± 6 and is tangent to the line at the point $(2,8)$.

a. The solid obtained by revolving $\mathscr{R}$ about the x axis consists of annuli of inner radius $9-\frac{1}{4}x^2$ and outer radius $10-x$, for $-6\leqslant x\leqslant 2$, so its volume is equal to

$$\pi\int_{-6}^{2}\left\{(10-x)^2-\left(9-\tfrac{1}{4}x^2\right)^2\right\}dx.$$

b. Let $\eta=10-\frac{1}{4}x^2$ if $2\leqslant x\leqslant 6$, and $\eta=1$ if $6<x\leqslant 10$. The solid obtained by revolving $\mathscr{S}$ about the line defined by $y=-1$ consists of annuli of inner radius η and outer radius $11-x$, for $2\leqslant x\leqslant 10$, so its volume is equal to

$$\pi\int_2^{10}\left\{(11-x)^2-\eta^2\right\}dx.$$

Alternatively, the right half of the parabola is defined by $x=\sqrt{36-4y}$, for $0\leqslant y\leqslant 9$, and the solid in question consists of cylindrical shells of radius $y+1$ and height $10-y-\sqrt{36-4y}$, for $0\leqslant y\leqslant 8$. So its volume is equal to

$$2\pi\int_0^8(y+1)\left(10-y-\sqrt{36-4y}\right)dy.$$

Solution to question 6. — By the mean value theorem, y is proportional to $\exp(z)$ on any interval where $\frac{dy}{dx}=y\frac{dz}{dx}$, for $\frac{d}{dx}(e^{-z}y)=e^{-z}(\frac{dy}{dx}-y\frac{dz}{dx})=0$. Therefore, if $y=-1$ when $x=0$ and

$$\frac{dy}{dx}=xy\sqrt{1+x^2},\qquad\text{then}\qquad y=-\exp\left\{\tfrac{1}{3}\left((1+x^2)^{3/2}-1\right)\right\}.$$

because $\frac{d}{dx}\left\{\frac{1}{3}(1+x^2)^{3/2}\right\}=x\sqrt{1+x^2}$.

Solution to question 7. — By the observation in the preceding solution, the number p of bacteria in the petri dish after t hours is proportional to $3^{t/2}$ (since it triples after two hours), and is thus given by $p=100\cdot 3^{t/2}$ (so that $p=100$ corresponds to $t=0$).

Solution to question 8. — If $x=\frac{1}{3}y^{3/2}-y^{1/2}$, then $\frac{dx}{dy}=\frac{1}{2}(\sqrt{y}-\sqrt{y}^{-1})$, so

$$1+\left(\frac{dx}{dy}\right)^2=1+\tfrac{1}{4}(\sqrt{y}-\sqrt{y}^{-1})^2=\tfrac{1}{4}(\sqrt{y}+\sqrt{y}^{-1})^2.$$

So the length of the curve is equal to

$$\frac{1}{2}\int_1^4(\sqrt{y}+\sqrt{y}^{-1})\,dy=\tfrac{1}{3}(y+3)\sqrt{y}\Big|_1^4=\tfrac{10}{3}.$$

Solution to question 9. — a. By inspecting dominant terms,

$$\lim_{n\to\infty}\arcsin\left(\frac{3n-2}{5-6n}\right)=\arcsin\left(-\tfrac{1}{2}\right)=-\tfrac{1}{6}\pi.$$

b. This sequence converges to zero, since

$$0<\int_{n+1}^{n+4}\frac{dx}{x+3}<\frac{3}{n+4}\qquad\text{if}\quad n\geqslant 0.$$

(So the term corresponding to $n\geqslant 0$ is >0 and $<\varepsilon$, provided $n>3/\varepsilon-4$).

Solution to question 10. — a. By the vanishing condition the sequence $\{a_n\}$ converges to 0, since the series $\sum a_n$ is convergent.

b. The vanishing condition implies that

$$\sum_{n=1}^{\infty} \frac{a_n - 1}{3 + a_n} \qquad \text{diverges, since} \qquad \lim \frac{a_n - 1}{3 + a_n} = -\tfrac{1}{3}$$

by part a.

c. As $0 < \vartheta < \tan(\vartheta)$ if $0 < \vartheta < \frac{1}{2}\pi$, it follows that $0 < \arctan(a_n) < a_n$ if $n \geqslant 1$. Since $\sum a_n$ is a convergent series, the comparison test implies that the series $\sum(-1)^n \arctan(a_n)$ is absolutely convergent.

Solution to question 11. — Since

$$\sum_{k=0}^{n-1} \{\arctan(k+1) - \arctan(k)\} = \arctan(n)$$

(by telescoping), the sum of the series is $\lim\limits_{n\to\infty} \arctan(n) = \frac{1}{2}\pi$.

Solution to question 12. — a. Since

$$0 < \frac{e^n + 3^n}{5^n - 2^n} < \frac{3^n + 3^n}{\frac{1}{2} \cdot 5^n} = 4\left(\tfrac{3}{5}\right)^n, \qquad \text{if} \quad n \geqslant 1,$$

and $\sum\left(\frac{3}{5}\right)^n$ is a convergent geometric series (its ratio, $\frac{3}{5}$, is positive and < 1), the comparison test implies that the series in question is convergent.

b. Since $0 < \sin(\vartheta) < \vartheta$ if $0 < \vartheta < \frac{1}{2}\pi$, it follows that

$$0 < \frac{\sin(1/n)}{n} < \frac{1}{n^2}, \qquad \text{if} \quad n \geqslant 1.$$

Therefore, the comparison test implies that the series in question converges with the p-series $\sum n^{-2}$ ($p = 2 > 1$).

Solution to question 13. — a. Resolving the absolute value of the general term into partial fractions gives

$$a_n = \frac{2n+1}{n(n+3)} = \frac{1}{3n} + \frac{5}{3(n+3)} \geqslant \frac{1}{3n} + \frac{5}{12n} = \frac{3}{4n}, \qquad \text{for} \quad n \geqslant 1.$$

So the comparison test implies that $\sum a_n$ diverges with the harmonic series. From the resolution into partial fractions of a_n it is plain that $0 < a_{n+1} < a_n$ and $\lim a_n = 0$, so the Leibniz test implies that $\sum(-1)^n a_n$ is convergent. Therefore, $\sum(-1)^n a_n$ is conditionally convergent.

b. If $n \geqslant 3$ then

$$a_n = \frac{(3n-3)!}{3n \cdot 3^{3n}} = \frac{1}{3^4} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdots \frac{3n-4}{3} \cdot \frac{3n-3}{3n} > \frac{2}{729}\left(1 - \frac{1}{n}\right)\left(n - \frac{4}{3}\right),$$

so the vanishing condition implies that $\sum(-1)^{n+1} a_n$ is divergent. (The ratio test could also be used, in which case ρ is ∞.)

c. If $n > 5$ then

$$0 < a_n = \left(\frac{n-5}{2n}\right)^n < \left(\frac{1}{2}\right)^n,$$

so the comparison test implies that $\sum a_n$ converges with the geometric series $\sum 2^{-n}$ (whose ratio, $\frac{1}{2}$, is positive and < 1). Therefore, $\sum(-1)^n a_n$ is absolutely convergent. (The root test could also be used, in which case ρ is $\frac{1}{2}$.)

Solution to question 14. — If $n > 1$, $x \neq -3$ and

$$\alpha_n = \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{3^n (n+1)!}(x+3)^n, \qquad \text{then} \qquad \left|\frac{\alpha_n}{\alpha_{n-1}}\right| = \tfrac{2}{3}|x+3| \cdot \frac{n + \frac{1}{2}}{n+1},$$

so the ratio test implies that $\sum \alpha_n$ is absolutely convergent if $\frac{2}{3}|x+3| < 1$, *i.e.*, $-\frac{9}{2} < x < -\frac{3}{2}$, and $\sum \alpha_n$ is divergent if $x < -\frac{9}{2}$ or $x > -\frac{3}{2}$. The radius of convergence of $\sum \alpha_n$ is $\frac{3}{2}$. Let a_n be the result of replacing x by $-\frac{3}{2}$ in α_n. Since

$$2n+2 > \sqrt{(2n+1)(2n+3)} \qquad \text{and} \qquad 2n+1 > 2\sqrt{n(n+1)}$$

(each of which is seen by squaring), it follows that if $n \geqslant 1$ then

$$\frac{\sqrt{n}}{\sqrt{n+1}} < \frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}, \qquad \text{and hence} \qquad \frac{1}{\sqrt{n+1}} < a_n < \frac{\sqrt{3}}{\sqrt{2n+3}},$$

for the middle term of the left inequality is a_1 if $n = 1$, and is a_n/a_{n-1} if $n > 1$. The comparison test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$ ($p = \frac{1}{2} \leqslant 1$), since $a_n > (n+1)^{-1/2} > \frac{1}{2}\sqrt{2}\, n^{-1/2}$ if $n \geqslant 1$. If $n > 1$ then $0 < a_n/a_{n-1} < 1$, or $0 < a_n < a_{n-1}$, and the right inequality of the last display implies that $\lim a_n = 0$, so the Leibniz test implies that $\sum(-1)^n a_n$ is convergent. Since $(-1)^n a_n$ is obtained by replacing x by $-\frac{9}{2}$ in α_n, the interval of convergence of $\sum \alpha_n$ is $\left[-\frac{9}{2}, -\frac{3}{2}\right)$.

Solution to question 15. — Since $\log(1+r) = \sum\limits_{k=1}^{\infty} (-1)^{k-1} r^k/k$ for $-1 < r \leqslant 1$, and $\log(x+3) = \log(5 + x - 2) = \log(5) + \log\left(1 + \frac{1}{5}(x-2)\right)$, it follows that

$$\log(x+3) = \log(5) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k 5^k}(x-2)^k,$$

provided $-1 < \frac{1}{5}(x-2) \leqslant 1$, *i.e.*, $-3 < x \leqslant 7$.