1. Evaluate the following integrals.

a. $\displaystyle\int \frac{3x^2+7x-3}{x^2(x-3)}\,dx$ b. $\displaystyle\int x\sin(x)\cos^2(x)\,dx$

c. $\displaystyle\int x^5\sqrt{x^3+1}\,dx$ d. $\displaystyle\int_{-3}^{1} \frac{3x}{\sqrt{55-x^2-6x}}\,dx$

e. $\displaystyle\int \frac{dx}{x\big((\log x)^2-25\big)^{3/2}}$ f. $\displaystyle\int \sqrt{1+e^x}\,dx$

2. Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$ b. $\displaystyle\lim_{x\to \pi^+}\left(\frac{2x-\pi}{\pi}\right)^{2\csc(x)}$

3. Evaluate each improper integral or show that it diverges.

a. $\displaystyle\int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx$ b. $\displaystyle\int_1^\infty x^2\log(x^2)\,dx$

4. Let $\mathscr{R}$ be the region bounded by the graphs of $y=x^3$ and $y=4x$. Write an integral which is equal to the volume of the solid obtained by revolving $\mathscr{R}$ about: a. the x-axis; b. the line defined by $x=3$.

5. Write an integral which is equal to the area of the region enclosed by the curves defined by $x=y^4$, $3y^2=x-4$ and $y\geqslant 0$.

6. Find the solution of the differential equation

$$\frac{dy}{dx}=\frac{xe^{x^2}}{y^2},$$

which satisfies the initial condition $y(0)=1$.

7. The Springfield pond has a volume of 10 ML (megalitres) pure, clear water. Mr. Burns begins dumping contaminated water, containing 20 g/ML of plutonium, at a rate of 5 ML per year. In order to not get caught, he also drains the pond at the same rate. Assuming the pond is kept thoroughly mixed, express the mass of plutonium in the pond as a function of time.

8. Determine whether the each sequence is convergent or divergent. If a sequence converges, compute its limit.

a. $\displaystyle\left\{\frac{n^2(3n-1)!}{(3n+1)!}\right\}$ b. $\displaystyle\left\{n\sin\!\left(\frac{1}{n}\right)\right\}$

9. Given that $a_1+a_2+a_3+\cdots+a_n=e^{2/n}-1$, compute each of the following.

a. $\displaystyle\sum_{n=1}^{\infty}a_n$ b. $\displaystyle\lim_{n\to\infty}a_n$ c. $\displaystyle\sum_{n=4}^{\infty}a_n$

10. Determine whether each series is convergent or divergent. Justify your answers completely.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{e^n+7^n}{n^2}$ b. $\displaystyle\sum_{k=1}^{\infty}\frac{k+5}{5^k}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{3\cos^2(n)}{\sqrt[3]{n+n^5}}$ d. $\displaystyle\sum_{m=1}^{\infty}\frac{7+3m}{\sqrt{5m^3+3}}$

11. Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your answers completely.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n^{3/2}}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(5n)5^n}{(5n)!}$ c. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{n+1}\right)^{n^2}$

12. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{5^n\sqrt{3n-1}}.$$

13. Find the Maclaurin series of xe^x.

14. Suppose that

$$\sum_{n=1}^{\infty}a_n \quad\text{and}\quad \sum_{n=1}^{\infty}b_n$$

are convergent series of positive terms. Either prove that the series

$$\sum_{n=1}^{\infty}a_nb_n$$

must be convergent, or else give an example to show that it may diverge.

1. a. Resolving the integrand into partial fractions gives
$$\frac{3x^2+7x-3}{x^2(x-3)} = -\frac{2}{x} + \frac{1}{x^2} + \frac{5}{x-3},$$
where the last two coefficients are obtained by inspection (covering and evaluating) and the first coefficient is then obtained by comparing quadratic coefficients of the combined numerator. Therefore,
$$\int \frac{3x^2+7x-3}{x^2(x-3)}\,dx = \log\left|\frac{(x-3)^5}{x^2}\right| - \frac{1}{x} + a.$$
b. Partial integration (integrating the trigonometric factors) gives
$$\int x\sin(x)\cos^2(x)\,dx = -\tfrac{1}{3}x\cos^3(x) + \tfrac{1}{3}\int \cos^3(x)\,dx.$$
If $c = \cos(x)$ and $s = \sin(x)$ then $c^3\,dx = (1-s^2)c\,dx = (1-s^2)\,ds$, so
$$\int \cos^3(x)\,dx = \int (1-s^2)\,ds = \sin(x) - \tfrac{1}{3}\sin^3(x) + b.$$
Therefore,
$$\int x\sin(x)\cos^2(x)\,dx = -\tfrac{1}{3}x\cos^3(x) + \tfrac{1}{3}\sin(x) - \tfrac{1}{9}\sin^3(x) + b.$$
c. If $y = \sqrt{x^3+1}$, or $y^2 = x^3+1$, then $2y\,dy = 3x^2\,dx$ and $x^3 = y^2-1$, so
$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= \tfrac{2}{3}\int (y^2-1)y^2\,dy = \tfrac{2}{15}y^5 - \tfrac{2}{9}y^3 + c \\ &= \tfrac{2}{45}(3y^2-5)y^3 + c = \tfrac{2}{45}(3x^3-2)\sqrt{(x^3+1)^3} + c.\end{aligned}$$
d. Since $55 - x^2 - 6x = 64 - (x+3)^2$ and $3x = 3(x+3) - 9$, it follows that
$$\begin{aligned}\int_{-3}^{1} \frac{3x}{\sqrt{55-x^2-6x}}\,dx &= -3\sqrt{55-x^2-6x}\,\Big|_{-3}^{1} - 9\arcsin\left(\tfrac{1}{8}(x+3)\right)\Big|_{-3}^{1} \\ &= -3(\sqrt{48}-8) - 9\left(\tfrac{1}{6}\pi\right) = 24 - 12\sqrt{3} - \tfrac{3}{2}\pi.\end{aligned}$$
e. If
$$y = \frac{\sqrt{(\log x)^2 - 25}}{\log(x)}, \quad \text{then} \quad y^2 = 1 - \frac{25}{(\log x)^2}, \quad \text{so} \quad \frac{y\,dy}{25} = \frac{dx}{x(\log x)^3}.$$
Thus, the integral in question is equal to
$$\begin{aligned}\int \left(\frac{\log(x)}{\sqrt{(\log x)^2-25}}\right)^3 \cdot \frac{dx}{x(\log x)^3} &= \int \frac{dy}{25y^2} = -\frac{1}{25y} + E \\ &= -\frac{\log(x)}{25\sqrt{(\log x)^2-25}} + E.\end{aligned}$$
f. If $y = \sqrt{1+e^x}$, then $x = \log(y^2-1)$ and $dx = 2y(y^2-1)^{-1}\,dx$, and so
$$\begin{aligned}\int \sqrt{1+e^x}\,dx &= \int \frac{2y^2}{y^2-1}\,dy = \int\left\{2 + \frac{2}{y^2-1}\right\}dy = 2y + \log\left(\frac{y-1}{y+1}\right) + F \\ &= \sqrt{1+e^x} + \log\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + F.\end{aligned}$$
2. a. Combining terms and expanding the exponential function gives
$$\lim_{x\to 0^+} \frac{1+x-e^x}{x(e^x-1)} = \lim_{x\to 0^+} \frac{-\frac{1}{2}x^2 - \frac{1}{6}x^3 + \&c.}{x^2 + \frac{1}{2}x^3 + \&c.} = -\tfrac{1}{2}.$$
b. If $x = \pi - y$ then $y \to 0^-$ as $x \to \pi^+$ and $\sin(x) = \sin(y)$, so
$$\lim_{x\to\pi^+} \left(\frac{2x-\pi}{\pi}\right)^{2\csc(x)} = \lim_{y\to 0^-}\left(1 - \frac{2y}{\pi}\right)^{-\frac{\pi}{2y}\cdot\left\{-\frac{4}{\pi}\cdot\frac{y}{\sin(y)}\right\}} = e^{-4/\pi}.$$
3. a. The integral is improper because the integrand is unbounded as $x \to 3^+$; integrating by inspection gives
$$\int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx = \lim_{\alpha\to 3^+} \sqrt{x^2-9}\,\Big|_{\alpha}^{5} = \lim_{\alpha\to 3^+}\left\{4 - \sqrt{\alpha^2-9}\right\} = 4.$$
b. The integral diverges to ∞ by comparison, since $x^2\log(x^2)$ is continuous on $[1,\infty)$, $x^2\log(x^2) > x^2$ if $x > \sqrt{e}$, and
$$\int_1^\infty x^2\,dx = \lim_{\beta\to\infty} \tfrac{1}{3}x^3\,\Big|_1^\beta = \infty.$$
4. The curves intersect where $0 = x^3 - 4x = x(x^2-4)$; *i.e.*, $x = 0, \pm 2$. Also, $|x^3| \leqslant |4x|$ provided $|x| \leqslant 2$, so $\mathscr{R} = \{(x,y):\ -2 \leqslant x \leqslant 2 \text{ and } |x^3| \leqslant y \leqslant |4x|\}$. The solid obtained by revolving $\mathscr{R}$ about the x-axis consists of annuli of inner radius $|x^3|$ and outer radius $|4x|$, for $-2 \leqslant x \leqslant 2$, so its volume is equal to
$$\pi\int_{-2}^{2}\left((4x)^2 - x^6\right)dx = 2\pi\int_0^2\left((4x)^2 - x^6\right)dx.$$
The solid obtained by revolving $\mathscr{R}$ about the line defined by $x = 3$ consists of concentric cylinders of radius $3 - x$ and height $|4x| - |x^3| = |4x - x^3|$, for $-2 \leqslant x \leqslant 2$, so its volume is equal to
$$2\pi\int_{-2}^{2}(3-x)|4x - x^3|\,dx.$$
5. The curves meet where $0 = y^4 - 3y^2 - 4 = (y^2-4)(y^2+1)$; *i.e.*, where $y = 0, \pm 2$. If $0 \leqslant y \leqslant 2$ then $y^4 \leqslant 3y^2 + 4$, so the area of the region in question is equal to (it is not necessary to give the second form)
$$\int_0^2 |y^4 - 3y^2 - 4|\,dy = \int_0^2 (4 + 3y^2 - y^4)\,dy.$$
6. Separating the variables (and multiplying by six) gives
$$6y^2\frac{dy}{dx} = 6xe^{x^2}, \qquad \text{so} \qquad 2y^3 = 3e^{x^2} - 2,$$
where the last term on right right side insures that $y = 1$ if $x = 0$. An explicit solution was not required, but it is easy to supply: $y = \left(\frac{1}{2}(3e^x - 2)\right)^{1/3}$.

7. If p denotes the mass (in grams) plutonium in the pond t years then
$$\frac{dp}{dt} = 20\cdot 5 - \tfrac{1}{10}p\cdot 5, \qquad \text{or} \qquad \frac{d}{dt}(p - 200) = -\tfrac{1}{2}(p-200),$$
which is an exponential equation, so $p - 200 = Ae^{-t/2}$ for some real number A (by the mean value theorem). Initially there is no plutonium in the pond, so $A = -200$, and thus $p = 200(1 - e^{-t/2})$.

8. a. Simplifying the factorials gives
$$\frac{n^2(3n-1)!}{(3n+1)!} = \frac{n^2}{(3n+1)(3n)} \to \tfrac{1}{9}, \qquad \text{as} \quad n \to \infty.$$
b. Since $x = 1/n \to 0^+$ as $n \to \infty$, it follows that
$$\lim\left\{n\sin(1/n)\right\} = \lim_{x\to 0^+}\frac{\sin(x)}{x} = 1.$$
9. a. By the orthodox definition of the sum of a series,
$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}(a_1 + a_2 + a_3 + \cdots + a_n) = \lim_{n\to\infty}\left(e^{2/n} - 1\right) = 0.$$
b. Since $\sum a_n$ is convergent, the vanishing condition implies that $\lim a_n = 0$.

c. By part a and the orthodox definition of the sum of a series,
$$\begin{aligned}\sum_{n=4}^{\infty} a_n &= \lim_{n\to\infty}(a_1 + a_2 + a_3 + \cdots + a_n) - (a_1 + a_2 + a_3) \\ &= 0 - \left(e^{2/3} - 1\right) = 1 - e^{2/3}.\end{aligned}$$
10. a. If $n \geqslant 1$ then
$$n^2 < e^n < 7^n, \qquad \text{and hence} \qquad \frac{e^n + 7^n}{n^2} > \frac{2e^n}{n^2} > 2,$$
so the series in question diverges by the vanishing condition.

b. If $k \geqslant 2$ then $k+5 < 4k < 4 \cdot 2^k$, so

$$0 < \frac{k+5}{5^k} < 4\left(\frac{2}{5}\right)^k.$$

Therefore, the comparison test implies that the series in question converges with the geometric series $\sum\left(\frac{2}{5}\right)^k$ (whose ratio, $\frac{2}{5}$, is > 0 and < 1).

c. If $n \geqslant 1$ then

$$0 < \frac{3\cos^2(n)}{\sqrt[3]{n+n^5}} < \frac{3}{n^{5/3}},$$

so the comparison test implies that the series in question converges with the p-series $\sum n^{-5/3}$ (for which $p = \frac{5}{3} > 1$).

d. If $m \geqslant 1$ then $7 + 3m > 3m$ and $5m^3 + 3 < 9m^3$, and hence

$$\frac{7+3m}{\sqrt{5m^3+3}} > \frac{3m}{\sqrt{9m^3}} = \frac{1}{\sqrt{m}} > 0;$$

so the comparison test implies that the series in question diverges with the p-series $\sum m^{-1/2}$ (for which $p = \frac{1}{2} \leqslant 1$).

11. a. If $n \geqslant 1$ and

$$a_n = \frac{2n+1}{n^{3/2}}, \qquad \text{then} \qquad a_n > \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}},$$

so the comparison test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$ (for which $p = \frac{1}{2} \leqslant 1$). Hence, $\sum(-1)^n a_n$ is not absolutely convergent. On the other hand, if $n \geqslant 1$ then

$$a_n = \frac{2}{\sqrt{n}} + \frac{1}{n^{3/2}} > \frac{2}{\sqrt{n+1}} + \frac{1}{(n+1)^{3/2}} = a_{n+1},$$

and $\lim a_n = 0$, so the Leibniz test implies that $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum(-1)^n a_n$ is conditionally convergent.

b. Notice that if $n \geqslant 2$ then $n^2 - n \geqslant \frac{1}{2}n^2$, and hence

$$0 < b_n = \frac{(5n)5^n}{(5n)!} = \frac{5^n}{(5n-1)!} = \frac{5}{5n-1} \cdot \frac{5}{5n-2} \cdots \frac{5}{4n} \cdot \frac{1}{4n-1} \cdots \frac{1}{2} \cdot \frac{1}{1} < \frac{5}{8n^2},$$

so the comparison test implies that $\sum b_n$ converges with the p-series $\sum n^{-2}$ (for which $p = 2 > 1$). Thus, the series $\sum(-1)^n b_n$ is absolutely convergent. (The ratio test could also be used.)

c. If

$$c_n = \left(\frac{n}{n+1}\right)^{n^2}, \qquad \text{then} \qquad \lim \sqrt[n]{c_n} = \lim\left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1,$$

so the series $\sum c_n$ converges by the root test. Therefore, the series $\sum(-1)^n c_n$ is absolutely convergent.

12. If $x \neq 1$ and

$$\alpha_n = (-1)^{n+1}\frac{(x-1)^n}{5^n\sqrt{3n-1}},$$

then

$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \tfrac{1}{5}|x-1| \lim\sqrt{\frac{3n-1}{3n+2}} = \tfrac{1}{5}|x-1|,$$

so the ratio test implies that $\sum \alpha_n$ is absolutely convergent if $\frac{1}{5}|x-1| < 1$, *i.e.*, $-4 < x < 6$, and is divergent if $x < -4$ or $x > 6$. The radius of convergence of $\sum \alpha_n$ is 5. If $n \geqslant 1$ and

$$a_n = \frac{1}{\sqrt{3n-1}}, \qquad \text{then} \qquad a_n > \frac{1}{\sqrt{3n}} > \frac{1}{\sqrt{3n+2}} = a_{n+1};$$

so the comparison test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$ (for which $p = \frac{1}{2} \leqslant 1$) and, since $\lim a_n = 0$, the Leibniz test implies that $\sum(-1)^{n+1} a_n$ is convergent. Since $\alpha_n = -a_n$ if $x = -4$ and $\alpha_n = (-1)^{n+1} a_n$ if $x = 6$, it follows that the interval of convergence of $\sum \alpha_n$ is $(-4, 6\,]$.

13. Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \text{it follows that} \qquad xe^x = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!},$$

for all real numbers x.

14. Since $\sum a_n$ is convergent and $a_n, b_n > 0$, there is a positive integer n_0 such that $0 < a_n < 1$, and thus $0 < a_n b_n < b_n$, for $n \geqslant n_0$; so the comparison test implies that $\sum a_n b_n$ converges with $\sum b_n$.