1. Evaluate the following integrals.

a. $\int x^2\cos^2(x^3)\,dx$ b. $\int e^{3x}\sin(2x)\,dx$

c. $\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}\,dx$ d. $\int \frac{dx}{x^4\sqrt{x^2-9}}$

e. $\int_0^{\frac{1}{4}\pi} 4\sec^4(\vartheta)\tan(\vartheta)\,d\vartheta$ f. $\int_1^{16}\frac{dx}{(1+\sqrt[4]{x})\sqrt{x}}$ g. $\int \frac{\log(2x)}{x\log(x)}\,dx$

2. Evaluate the following improper integrals.

a. $\int_0^1 \frac{\log(x)}{\sqrt{x}}\,dx$ b. $\int_{-\infty}^{\infty}\frac{dx}{x^2+9}$

3. Evaluate the following limits.

a. $\lim_{x\to\frac{1}{2}\pi}\frac{1-\sin(x)}{(2x-\pi)^2}$ b. $\lim_{x\to 2}\left(\sin(\pi/x)\right)^{\tan(\pi/x)}$

4. Let $\mathscr{R}$ be the region bounded by the graphs of $y=\arcsin(x)$, $y=0$ and $x=1$.

a. Evaluate the area of $\mathscr{R}$.

b. Write an integral which represents the volume of the solid obtained by revolving $\mathscr{R}$ about the line defined by $y=\frac{1}{2}\pi$.

c. Write an integral which represents the volume of the solid obtained by revolving $\mathscr{R}$ about the line defined by $x=2$.

5. Solve the differential equation

$$x\sqrt{1-y^2}-\sqrt{1-x^2}\,\frac{dy}{dx}=0,$$

given that $y(1)=\frac{1}{2}\sqrt{3}$. Express y as a(n explicit) function of x.

6. Find an equation of the curve which passes through the point $(-2,e)$ that has the property that the slope of the tangent line at any of its points is equal to the product of the x and y coordinates of that point.

7. Determine whether the sequence $\{a_n\}$ converges or diverges. If the sequence converges find its limit; otherwise, explain why it diverges.

a. $a_n=(-1)^n\frac{\sqrt{n+3}}{5-3\sqrt{n}}$ b. $a_n=n^2\cos(1/n)-n^2$

8. Find the sum of the series

$$\sum_{n=2}^{\infty}\frac{\pi+(-2)^n}{3^n}.$$

9. Determine whether the series converges or diverges. Justify all assertions carefully.

a. $\frac{1}{3}-\cos\left(\frac{1}{3}\right)+\frac{1}{9}-\cos\left(\frac{1}{9}\right)+\frac{1}{27}-\cos\left(\frac{1}{27}\right)+\frac{1}{81}-\cos\left(\frac{1}{81}\right)+\cdots+3^{-n}-\cos\left(3^{-n}\right)+\cdots$

b. $\sum_{n=1}^{\infty}\frac{\arctan(n)}{\sqrt[4]{n^9+8}}$ c. $\sum_{n=1}^{\infty}\log(1+1/n)$

10. Determine whether each of the following series is absolutely convergent, conditionally convergent or divergent. Justify all assertions carefully.

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{3^n n^n}$ b. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+2}+n}$

11. Find the interval of convergence of the power series

$$\sum_{n=2}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n\log(n)}.$$

12. Find the Taylor series of $\cos(x)$ centred at $\frac{1}{2}\pi$.

13. Suppose that the power series

$$f(x)=\sum_{n=1}^{\infty}c_n(x-2)^n$$

converges if $x=-2$ and diverges if $x=-3$.

a. Does the series converge if $x=6$, or does it diverge, or could it either converge or diverge? Explain.

b. Does the series $\sum c_n$ converge, or does it diverge, or could it converge or diverge? Explain.

c. Show that the series $\sum nc_n$ converges.

1. a. If $y = x^3$ then $dy = 3x^2\,dx$, so that

$$\int x^2\cos^2(x^3)\,dx = \tfrac{1}{3}\int \cos^2(y)\,dy = \tfrac{1}{6}\int\Big(1+\sin(2y)\Big)dy = \tfrac{1}{6}x^3 + \tfrac{1}{12}\sin(2x^3) + a.$$

b. Repeated partial integration (integrating the exponential factor) and absorption gives

$$\int e^{3x}\sin(2x)\,dx = \tfrac{1}{3}e^{3x}\sin(2x) - \tfrac{2}{9}e^{3x}\cos(2x) - \tfrac{4}{9}\int e^{3x}\sin(2x)\,dx = \tfrac{1}{13}e^{3x}\Big(3\sin(2x) - 2\cos(2x)\Big) + b.$$

c. Resolving the integrand into partial fractions, and then revising the second term gives

$$\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}\,dx = \int\left\{\frac{-1}{x-1} + \frac{3x-2}{x^2-2x+5}\right\}dx = \int\left\{\frac{-1}{x-1} + \frac{3x-3}{x^2-2x+5} + \frac{1}{(x-1)^2+4}\right\}dx$$
$$= -\log|x-1| + \tfrac{3}{2}\log(x^2-2x+5) + \tfrac{1}{2}\arctan\Big(\tfrac{1}{2}(x-1)\Big) + c.$$

d. If $y = x^{-1}\sqrt{x^2-9}$ then $y^2 = 1 - 9x^{-2}$, $y\,dy = 9x^{-3}\,dx$ and $9x^{-2} = 1 - y^2$, so

$$\int \frac{dx}{x^4\sqrt{x^2-9}} = \frac{1}{81}\int \frac{9}{x^2}\cdot\frac{x}{\sqrt{x^2-9}}\cdot\frac{9}{x^3}\,dx = \int(1-y^2)\cdot\frac{1}{y}\cdot y\,dy$$
$$= \tfrac{1}{81}\Big(y - \tfrac{1}{3}y^3\Big) + d = \frac{(3-y^2)y}{243} + d = \frac{(2x^2+9)\sqrt{x^2-9}}{243x^3} + d,$$

since $3 - y^2 = 2 + 9x^{-2} = (2x^2+9)/x^2$.

e. If $y = \sec(\vartheta)$ then $dy = \sec(\vartheta)\tan(\vartheta)\,d\vartheta$, $y = 1$ if $\vartheta = 0$ and $y = \sqrt{2}$ if $\vartheta = \frac{1}{4}\pi$, so that

$$\int_0^{\frac{1}{4}\pi} 4\sec^4(\vartheta)\tan(\vartheta)\,d\vartheta = \int_1^{\sqrt{2}} 4y^3\,dy = y^4\Big|_1^{\sqrt{2}} = 3.$$

f. If $y = 1 + \sqrt[4]{x}$ then $x = (y-1)^4$, $dx = 4(y-1)^3\,dy$ and $\sqrt{x} = (y-1)^2$. Also, $y = 2$ if $x = 1$ and $y = 3$ if $x = 16$, so

$$\int_1^{16}\int \frac{dx}{(1+\sqrt[4]{x})\sqrt{x}} = 4\int_2^3 \frac{(y-1)^3}{y(y-1)^2}\,dy = 4\int_2^3 (1 - 1/y)\,dy = 4(y - \log(y))\Big|_2^3 = 4\log\Big(\tfrac{2}{3}e\Big).$$

g. If $y = \log(x)$ then $dy = x^{-1}\,dx$ and $\log(2x) = y + \log(2)$, so

$$\int \frac{\log(2x)}{x\log(x)}\,dx = \int \frac{y+\log(2)}{y}\,dy = y + \log(2)\log|y| + g = \log(x) + \log(2)\log|\log(x)| + g.$$

2. a. Partial integration gives

$$\int_0^1 \frac{\log(x)}{\sqrt{x}}\,dx = \lim_{\varepsilon\to0^+} 2\sqrt{x}\log(x)\Big|_\varepsilon^1 - 2\int_0^1 \frac{dx}{\sqrt{x}} = -4\lim_{\varepsilon\to0^+}\sqrt{x}\Big|_\varepsilon^1 = -4,$$

where the first limit is zero by elementary properties of the logarithm.

b. Using symmetry and a standard integral,

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+9} = 2\int_0^\infty \frac{dx}{x^2+9} = \tfrac{2}{3}\lim_{t\to\infty}\arctan\Big(\tfrac{1}{3}x\Big)\Big|_0^t = \tfrac{1}{3}\pi.$$

3. a. If $y = \frac{1}{2}\pi - x$ then, since $\cos(y) = 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 - \&c.$, it follows that

$$\lim_{x\to\frac{1}{2}\pi}\frac{1-\sin(x)}{(2x-\pi)^2} = \lim_{y\to0}\frac{1-\cos(y)}{4y^2} = \tfrac{1}{8}.$$

b. If $y = \sin(\pi/x)$ then

$$\lim_{x\to2}\Big(\sin(\pi/x)\Big)^{\tan(\pi/x)} = \lim_{y\to1} y^{\frac{y}{\sqrt{1-y^2}}} = \lim_{y\to1}(1+y-1)^{\frac{1}{y-1}\cdot\frac{-y\sqrt{1-y}}{\sqrt{1+y}}} = e^0 = 1.$$

4. Observe that $\mathscr{R} = \{(x,y)\colon 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant \arcsin(x)\}$, or equivalently, $\mathscr{R} = \{(x,y)\colon 0\leqslant y\leqslant \frac{1}{2}\pi,\ 0\leqslant x\leqslant\sin(y)\}$.

a. The area of $\mathscr{R}$ is equal to

$$\int_0^1 \arcsin(x)\,dx = x\arcsin(x)\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \tfrac{1}{2}\pi + \sqrt{1-x^2}\Big|_0^1 = \tfrac{1}{2}\pi - 1,$$

by partial integration and inspection. Alternatively, the area of $\mathscr{R}$ is equal to

$$\int_0^{\frac{1}{2}\pi}(1-\sin(y))\,dy = (y+\cos(y))\Big|_0^{\frac{1}{2}\pi} = \tfrac{1}{2}\pi - 1.$$

b. The solid obtained by revolving $\mathscr{R}$ about the line defined by $y = \frac{1}{2}\pi$ consists of annuli of outer radius $\frac{1}{2}\pi$ and inner radius $\frac{1}{2}\pi - \arcsin(x)$, for $0\leqslant x\leqslant 1$, so its volume is equal to

$$\pi\int_0^1\Big\{(\tfrac{1}{2}\pi)^2 - (\tfrac{1}{2}\pi - \arcsin(x))^2\Big\}dx.$$

Alternatively, the solid consists of concentric cylindrical shells of radius $\frac{1}{2}\pi - y$ and height $1 - \sin(y)$, for $0\leqslant y\leqslant\frac{1}{2}\pi$, so its volume is equal to

$$2\pi\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi - y)(1-\sin(y))\,dy.$$

c. The solid obtained by revolving $\mathscr{R}$ about the line defined by $x = 2$ consists of concentric cylindrical shells of radius $2 - x$ and height $\arcsin(x)$, for $0\leqslant x\leqslant 1$, so its volume is equal to

$$2\pi\int_0^1 (2-x)\arcsin(x)\,dx.$$

Alternatively, the solid consists of annuli of inner radius 1 and outer radius $2 - \sin(y)$, for $0\leqslant y\leqslant\frac{1}{2}\pi$, so its volume is equal to

$$\pi\int_0^{\frac{1}{2}\pi}\Big\{(2-\sin(y))^2 - 1\Big\}dy.$$

5. Separating the variables and integrating gives

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}, \qquad\text{and hence}\qquad \arcsin(y) = a - \sqrt{1-x^2},$$

in which $a = \frac{1}{3}\pi$ since $y(1) = \frac{1}{2}\sqrt{3}$. Therefore, $y = \sin\Big(\frac{1}{3}\pi - \sqrt{1-x^2}\Big)$.

6. The condition on the slope of the tangent line is equivalent to

$$\frac{dy}{dx} = xy,\quad \textit{i.e.},\quad e^{-\frac{1}{2}x^2}\frac{dy}{dx} - e^{-\frac{1}{2}x^2}xy = 0 \quad\text{or}\quad \frac{d}{dx}\Big\{e^{-\frac{1}{2}x^2}y\Big\} = 0,$$

Thus, $e^{-\frac{1}{2}x^2}y$ is constant, and therefore $e^{-\frac{1}{2}x^2}y = e^{-\frac{1}{2}(-2)^2}e = e^{-1}$, using the initial condition. So the curve is defind by $e^{-\frac{1}{2}x^2}y = e^{-1}$, or $y = e^{\frac{1}{2}x^2-1}$.

7. a. Since

$$\lim\frac{\sqrt{n+3}}{5-3\sqrt{n}} = -\tfrac{1}{3}, \qquad\text{it follows that}\qquad \left\{(-1)^n\frac{\sqrt{n+3}}{5-3\sqrt{n}}\right\}$$

(oscillates and) diverges.

b. With $x = 1/n$, that $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \&c.$, implies that

$$\lim a_n = \lim\Big\{n^2(\cos(1/n)-1)\Big\} = \lim_{x\to0}\frac{\cos(x)-1}{x^2} = \tfrac{1}{2}.$$

8. By a direct computation,

$$\sum_{n=2}^\infty\frac{\pi+(-2)^n}{3^n} = \pi\sum_{n=2}^\infty\Big(\tfrac{1}{3}\Big)^n + \sum_{n=2}^\infty\Big(-\tfrac{2}{3}\Big)^n = \frac{\pi}{9}\cdot\frac{1}{1-\frac{1}{3}} + \frac{4}{9}\cdot\frac{1}{1+\frac{2}{3}} = \tfrac{1}{6}\pi + \tfrac{4}{15}.$$

9. a. Since $\lim\cos(3^{-n}) \neq 0$ (and $\lim 3^n = 0$), the series in question diverges by the vanishing condition.

b. If $n \geqslant 1$ then

$$0 < \frac{\arctan(n)}{\sqrt[4]{n^9+8}} < \frac{\pi}{2n^{9/4}},$$

so the series in question converges with the p-series $\sum n^{-9/4}$ ($p = \frac{9}{4} > 1$) by the comparison test.

c. If $x > 1$ then $1 - 1/x < \log(x) < x - 1$, so in particular (taking $x = 1 + 1/n$)

$$\frac{1}{n+1} < \log(1+1/n) < 1/n \qquad \text{for} \quad n \geqslant 1.$$

Hence, the series in question diverges with the harmonic series (*i.e.*, the p-series with $p = 1$).

10. a. If $a_n = n!/(3^n n^n)$, then $a_n > 0$ if $n \geqslant 1$ and

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)!\,3^n n^n}{n!\,3^{n+1}(n+1)^{n+1}} = \lim \tfrac{1}{3}\left(1+\frac{1}{n}\right)^{-n} = \tfrac{1}{3}e^{-1} < 1,$$

so the ratio test implies that the series $\sum(-1)^n a_n$ is absolutely convergent.

b. Since

$$a_n = \frac{1}{\sqrt{n^2+2}+n} > \frac{1}{\sqrt{4n^2}+n} = \frac{1}{3n} > 0,$$

for $n \geqslant 1$, the series $\sum a_n$ diverges with the harmonic series. But if $n \geqslant 1$ then $\sqrt{(n+1)^2+2} + n + 1 > \sqrt{n^2+2} + n > 0$, and $\lim\left\{\sqrt{n^2+2}+n\right\} = \infty$, so $0 < a_{n+1} < a_n$ for $n \geqslant 1$ and $\lim a_n = 0$; Hence, the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum(-1)^n a_n$ is conditionally convergent.

11. If

$$\alpha_n = (-1)^{n+1}\frac{(x-2)^n}{n\log(n)}$$

and $x \neq 2$ then

$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = |x-2|\lim\left\{\frac{n}{n+1}\cdot\frac{\log(n)}{\log(n+1)}\right\} = |x-2|,$$

so the ratio test implies that the series $\sum\alpha_n$ is convergent if $|x-2| < 1$, *i.e.*, $-1 < x < 3$, and that $\sum\alpha_n$ is divergent if $x < -1$ or $x > 3$. If $x = -1$ then $\alpha_n = -a_n$, where $a_n = (n\log n)^{-1}$ and if $x = 3$ then $\alpha_n = (-1)^{n+1}\alpha_n$. Since $\sum a_n$ is a divergent logarithmic p-series ($p = 1$), the series $\sum\alpha_n$ diverges if $x = -1$. On the other hand, $(n+1)\log(n+1) > n\log(n) > 0$ if $n \geqslant 2$ and $\lim(n\log(n)) = \infty$, so $a_n > a_{n+1} > 0$ if $n \geqslant 2$ and $\lim a_n = 0$. Thus, the Leibniz test implies that $\sum(-1)^{n+1}a_n$ is convergent; *i.e.*, that $\sum\alpha_n$ converges if $x = 3$. Therefore, the radius of convergence of $\sum\alpha_n$ is 1, and its interval of convergence is $(-1, 3]$.

12. Since $\cos(x) = \cos(\frac{1}{2}\pi + x - \frac{1}{2}\pi) = -\sin(x - \frac{1}{2}\pi)$, the Maclaurin expansion of the sine function yields

$$\cos(x) = -\sin(x - \tfrac{1}{2}\pi) = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}(x - \tfrac{1}{2}\pi)^{2k+1}.$$

13. The interval of convergence of $f(x)$ contains the interval $[-2, 6)$, is contained in the interval $(-3, 7]$, and may be equal to either of these intervals.

a. The series $f(6)$ may or may not converge, as 6 does not belong to $[-2, 6)$.

b. The series $\sum c_n$ is $f(1)$, which converges, since 1 does belong to $[-2, 6)$.

c. As $|4-2| < |-2-2|$, and $f(-2)$ converges, $f(4) = \sum_{n=1}^{\infty} c_n 2^n$ is absolutely convergent. But if $n \geqslant 1$, then $0 < n < 2^n$, and hence $0 \leqslant |nc_n| \leqslant |c_n 2^n|$. So the comparison test implies that $\sum|nc_n|$ converges with $f(4)$, and therefore the series $\sum nc_n$ is (absolutely) convergent.