Question 1. — Evaluate each of the following integrals.

a. $\displaystyle\int \frac{5x^2-3x+10}{(x-2)(x^2+4)}\,dx$ b. $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{\sin^4(x)}{\cos^6(x)}\,dx$ c. $\displaystyle\int \frac{\sqrt{x^2-25}}{x^4}\,dx$

d. $\displaystyle\int e^{-x}\cos(3x)\,dx$ e. $\displaystyle\int_0^9 \frac{dx}{\sqrt{\sqrt{x}+1}}$ f. $\displaystyle\int_0^{\log(3)} \frac{e^x}{\sqrt{15+2e^x-e^{2x}}}\,dx$

g. $\displaystyle\int x\arcsin(x)\,dx$

Question 2. — Evaluate the following limits.

a. $\displaystyle\lim_{x\to\pi}\frac{\sin^2(2x)}{1+\cos(x)}$ b. $\displaystyle\lim_{x\to\infty}\left\{x\left(\tfrac{1}{2}\pi-\arctan(x)\right)\right\}$ c. $\displaystyle\lim_{x\to 0^+}\left(1+\tfrac{1}{2}x\right)^{4/x}$

Question 3. — Determine whether each integral converges or diverges. If an integral converges, give its exact value.

a. $\displaystyle\int_{-1}^{11}\frac{dx}{\sqrt[3]{(x-3)^4}}$ b. $\displaystyle\int_0^{\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Question 4. — Solve the differential equation

$$\sqrt{1-x^2}\,\frac{dy}{dx}=x+xy^2,$$

subject to the initial condition $y(0)=1$.

Question 5. — A rumour starts in a town with a population of 1000. The rumour spreads at a rate proportional to the number of people who at time t (in weeks) have not heard the rumour. Initially 25 people heard the rumour and after three weeks 675 people had heard it. How many people will have heard the rumour after six weeks?

Question 6. — Let $\mathscr{R}$ be the region bounded by the graphs of $y=\sin(x)$ and the x-axis, with $x\in[0,\pi]$. Sketch $\mathscr{R}$, and write an integral representing the volume of the solid obtained by revolving $\mathscr{R}$ about: a. the x-axis, b. the line defined by $x=-2$ and c. the line defined by $y=2$. Do not evaluate the integrals.

Question 7. — Find the length of the curve defined by

$$y=\arcsin(x)+\sqrt{1-x^2}.$$

Question 8. — a. Does the series $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{(2n)!}$ converge? If so, to what value?

b. Does the sequence $\displaystyle\left\{\frac{3^n}{(2n)!}\right\}$ converge? If so, to what value?

Question 9. — Determine whether the following series converge or diverge. State which tests you are using and display a proper solution.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan(n)}{\sqrt[3]{n^2+1}}$ b. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}-\frac{3}{2^{n-1}}\right)$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3n+n\cos^2(n)}$

Question 10. — Find the sum of the series

$$\sum_{n=1}^{\infty}\left\{\cos^{-1}\left(\frac{1}{n}\right)-\cos^{-1}\left(\frac{1}{n+1}\right)\right\}.$$

Question 11. — Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{e^{1/n^2}}{n}$ b. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{n+1}{3n+5}\right)^{2n}$

Question 12. — Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n+1}}{4^n}(x-3)^n.$$

Question 13. — Find the Maclaurin expansion of $\log(x+2)$. Express the series using sigma notation and write the first four non-zero terms of the series explicitly.

Solution to question 1. — a. The resolution into partial fractions is

$$\frac{5x^2-3x+10}{(x-2)(x^2+4)} = \frac{3}{x-2} + \frac{2x+1}{x^2+4},$$

where the first coefficient is obtained by covering, and the second and third coefficients are obtained by comparing quadratic and constant coefficients in the numerator. Therefore, the integral is equal to

$$\log|(x-2)^3(x^2+4)| + \tfrac{1}{2}\arctan(\tfrac{1}{2}x).$$

b. If $t = \tan(x)$ then $dt = \sec^2(x)\,dx$, and

$$\int_0^{\frac{1}{4}\pi} \frac{\sin^4(x)}{\cos^6(x)}\,dx = \int_0^{\frac{1}{4}\pi} \tan^4(x)\sec^2(x)\,dx = \int_0^1 t^4\,dt = \tfrac{1}{5}.$$

c. If $y = x^{-1}\sqrt{x^2-25}$, *i.e.*, $y^2 = 1-25x^{-2}$, then $y\,dy = 25x^{-3}\,dx$; therefore,

$$\int \frac{\sqrt{x^2-25}}{x^4}\,dx = \int \frac{\sqrt{x^2-25}}{x}\cdot\frac{dx}{x^3} = \tfrac{1}{25}\int y^2\,dy = \frac{(x^2-25)^{3/2}}{75x^3}.$$

d. Repeated partial integration, integrating the exponential factor, gives

$$\int e^{-x}\cos(3x)\,dx = -e^{-x}\cos(3x) + 3e^{-x}\sin(3x) - 9\int e^{-x}\cos(3x)\,dx$$
$$= \tfrac{1}{10}e^{-x}(3\sin(3x) - \cos(3x)),$$

by absorbing the integral at the right on the left side.

e. If $y = \sqrt{\sqrt{x}+1}$ then $x = (y^2-1)^2$, so $dx = 4y(y^2-1)\,dy$; therefore,

$$\int_0^9 \frac{dx}{\sqrt{\sqrt{x}+1}} = 4\int_1^2 (y^2-1)\,dy = 4\left(\tfrac{1}{3}y^3 - y\right)\Big|_1^2 = 4\left(\tfrac{7}{3}-1\right) = \tfrac{16}{3}.$$

f. Since $15+2e^x-e^{2x} = 16-(e^x-1)^2$, and $d(e^x-1) = e^x\,dx$, it follows that

$$\int_0^{\log(3)} \frac{e^x}{\sqrt{15+2e^x-e^{2x}}}\,dx = \arcsin\left(\tfrac{1}{4}(e^x-1)\right)\Big|_0^{\log(3)} = \tfrac{1}{6}\pi.$$

g. If $\vartheta = \arcsin(x)$ then $x\,dx = \sin(\vartheta)\cos(\vartheta)\,d\vartheta = \frac{1}{2}\sin(2\vartheta)\,d\vartheta$; partial integration then gives

$$\int x\arcsin(x)\,dx = \tfrac{1}{2}\int \vartheta\sin(2\vartheta)\,d\vartheta = -\tfrac{1}{4}\vartheta\cos(2\vartheta) + \tfrac{1}{8}\sin(2\vartheta)$$
$$= \tfrac{1}{4}(2x^2-1)\arcsin(x) + \tfrac{1}{4}x\sqrt{1-x^2},$$

as $\cos(2\vartheta) = 1-2\sin^2(\vartheta) = 1-2x^2$ and $\sin(2\vartheta) = 2\sin(\vartheta)\cos(\vartheta) = 2x\sqrt{1-x^2}$.

Solution to question 2. — a. As $\sin^2(2x) = 4\cos^2(x)(1-\cos(x))(1+\cos(x))$, it follows that

$$\lim_{x\to\pi} \frac{\sin^2(2x)}{1+\cos(x)} = \lim_{x\to\pi}\left(4\cos^2(x)(1-\cos(x))\right) = 8.$$

b. The limit is equal to

$$\lim_{x\to\infty} \frac{\frac{1}{2}\pi - \arctan(x)}{1/x} = \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2} = 1,$$

by one application of l'Hôpital's rule and inspection of the dominant terms.

c. Elementary properties of the logarithm give $\lim\limits_{t\to 0}(1+t)^{1/t} = e$, and hence

$$\lim_{x\to 0^+}\left(1+\frac{x}{2}\right)^{4/x} = \lim_{x\to 0^+}\left(1+\frac{x}{2}\right)^{\frac{2}{x}\cdot 2} = e^2.$$

Solution to question 3. — a. Since

$$\int_3^{11} \frac{dx}{(x-3)^{4/3}} = \lim_{\alpha\to 3^+}\left\{-3(x-3)^{-1/3}\right\}\Big|_\alpha^{11} = \infty,$$

the integral diverges (to ∞, for the integrand is positive where defined).

b. Integrating by inspection gives

$$\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+}\left\{-2e^{\sqrt{x}}\right\}\Big|_\varepsilon^1 + \lim_{\beta\to\infty}\left\{-2e^{-\sqrt{x}}\right\}\Big|_1^\beta = 2.$$

Solution to question 4. — Separating the variables and integrating gives

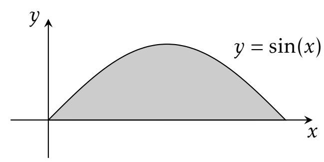
$$\frac{1}{1+y^2}\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}, \qquad \text{and so} \qquad \arctan(y) = \tfrac{1}{4}\pi + 1 - \sqrt{1-x^2},$$

since $y = 1$ if $x = 0$. Thus,

$$y = \tan\left(\tfrac{1}{4}\pi + 1 - \sqrt{1-x^2}\right) = \frac{1+\tan\left(1-\sqrt{1-x^2}\right)}{1-\tan\left(1-\sqrt{1-x^2}\right)}.$$

Solution to question 5. — If p denotes the number of people who have heard the rumour after t weeks, then $\frac{d}{dt}(1000-p) = \alpha(1000-p)$, for some real number α. This is an exponential differential equation, so the mean value theorem implies that $1000-p = 975a^t$, where $a = e^\alpha$, since $p = 25$ when $t = 0$. Next, since $p = 675$ when $t = 3$, it follows that $325 = 975a^3$, or $a = 3^{-1/3}$. If $t = 6$ then $1000-p = 975\cdot\frac{1}{9}$, or $p = 1000 - \frac{325}{3} = 1000 - 108\frac{1}{3} = 891\frac{2}{3}$, so after six weeks 891 people have heard the rumour.

Solution to question 6. — The region $\mathscr{R}$ is the shaded region in the figure below.



a. The solid obtained by revolving $\mathscr{R}$ about the x axis consists of disks of radius $\sin(x)$, for $0 \leqslant x \leqslant \pi$, so its volume is equal to

$$\pi\int_0^\pi \sin^2(x)\,dx.$$

b. The solid obtained by revolving $\mathscr{R}$ about the line defined by $x = -2$ consists of cylindrical shells of radius $x+2$ and height $\sin(x)$, so its volume is equal to

$$2\pi\int_0^\pi (x+2)\sin(x)\,dx.$$

c. The solid obtained by revolving $\mathscr{R}$ about the line defined by $y = 2$ consists of annuli of outer radius 2 and inner radius $2-\sin(x)$, so its volume is equal to

$$\pi\int_0^\pi \left\{4-(2-\sin(x))^2\right\}dx.$$

Solution to question 7. — If $y = \arcsin(x) + \sqrt{1-x^2}$ then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sqrt{\frac{1-x}{1+x}}, \qquad \text{so} \qquad 1+\left(\frac{dy}{dx}\right)^2 = \frac{2}{1+x}.$$

Therefore, the length of the curve is

$$\sqrt{2}\int_{-1}^1 \frac{dx}{\sqrt{1+x}} = 2\sqrt{2}\lim_{\alpha\to -1^+}\sqrt{1+x}\,\Big|_\alpha^1 = 4.$$

Solution to question 8. — a. If $n \geqslant 1$, then

$$0 < \frac{3^n}{(2n)!} = \frac{\sqrt{3}}{1}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{4}\cdots\frac{\sqrt{3}}{2n-1}\cdot\frac{\sqrt{3}}{2n} \leqslant \frac{3}{2n(2n-1)} \leqslant \frac{3}{2n^2},$$

so the comparison test implies that $\sum 3^n/(2n)!$ converges with $\sum n^{-2}$. In fact, since

$$\tfrac{1}{2}\left(e^x+e^{-x}\right) = \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!} = 1 + \sum_{n=1}^\infty \frac{x^{2n}}{(2n)!},$$

it follows that the sum of the series is $\frac{1}{2}\left(e^{\sqrt{3}}+e^{-\sqrt{3}}\right)-1$.

b. By part a, $\lim\left\{3^n/(2n)!\right\} = 0$ (directly, or by the vanishing condition).

Solution to question 9. — a. If $n \geqslant 1$ then

$$0 < \frac{\pi}{4n^{2/3}} < \frac{\arctan(n)}{\sqrt[3]{n^2+1}},$$

so the comparison test implies that the series in question diverges with the p-series $\sum n^{-2/3}$ ($p = \frac{2}{3} \leqslant 1$).

b. Inspecting dominant terms yields

$$\lim\left\{\frac{n^2+1}{2n^2+1} - \frac{3}{2^{n-1}}\right\} = \tfrac{1}{2},$$

so the vanishing condition implies that the series in question is divergent.

c. If $n \geqslant 1$ then $0 < \cos^2(n) < 1$, so

$$0 < \frac{1}{4n} < \frac{1}{3n + n\cos^2(n)},$$

so the comparison test implies that the series in question diverges with the harmonic series.

Solution to question 10. — If $A_n = \arccos(1/n)$, then $A_1 = 0$ and the sum of the first n terms of the series in question is equal to

$$A_1 - A_2 + A_2 - A_3 + \cdots + A_n - A_{n+1} = A_1 - A_{n+1} = -\arccos\left(\frac{1}{n+1}\right).$$

Since $\arccos(x) \to \frac{1}{2}\pi$ as $x \to 0$, the sum of the series is $-\frac{1}{2}\pi$.

Solution to question 11. — a. If $n \geqslant 1$ then $e^{1/n^2} > 1$, so $0 < 1/n < e^{1/n^2}/n$, and the comparision test implies that $\sum e^{1/n^2}/n$ diverges with the harmonic series. On the other hand, $e^{1/(n+1)^2} < e^{1/n^2}$ and $1/(n+1) < 1/n$ if $n \geqslant 1$, so the sequence $\{e^{1/n^2}/n\}$ is decreasing. Thus, since $\lim\{e^{1/n^2}/n\} = 0$, the Leibniz test implies that $\sum(-1)^n e^{1/n^2}/n$ is convergent. Therefore, the series $\sum(-1)^n e^{1/n^2}/n$ is conditionally convergent.

b. If $n \geqslant 1$ and

$$a_n = \left(\frac{n+1}{3n+5}\right)^{2n}, \qquad \text{then} \qquad 0 < a_n < \left(\frac{n+n}{3n}\right)^{2n} = \left(\tfrac{4}{9}\right)^n,$$

so the comparison test implies that $\sum a_n$ converges with the geometric series $\sum\left(\frac{4}{9}\right)^n$. Therefore, the series $\sum(-1)^n a_n$ is absolutely convergent.

Solution to question 12. — If $x \neq 3$ and

$$\alpha_n = (-1)^n \frac{\sqrt{n+1}}{4^n}(x-3)^n,$$

then

$$\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \tfrac{1}{4}|x-3| \lim \frac{\sqrt{n+2}}{\sqrt{n+1}} = \tfrac{1}{4}|x-3|,$$

so the ratio test implies that the series $\sum \alpha_n$ is absolutely convergent if $|x-3| < 4$, *i.e.*, $-1 < x < 7$, and is divergent if $x < -1$ or $x > 7$. If $x = 7$ or -1 and $n \geqslant 1$, then $\lim|\alpha_n| = \infty$ since $\alpha_n = (\pm 1)^n \sqrt{n+1}$; so $\sum \alpha_n$ diverges by the vanishing condition. Therefore, $\sum \alpha_n$ has radius of convergence 4 and interval of convergence $(-1, 7)$.

Solution to question 13. — The Maclaurin expansion of $\log(1+x)$, which is obtained by integrating the geometric series $1/(1+t) = \sum_{k \geqslant 0}(-1)^k t^k$, yields

$$\log(x+2) = \log(2) + \log\left(1 + \tfrac{1}{2}x\right) = \log(2) + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2^k k} x^k$$
$$= \log(2) + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{24}x^3 - \tfrac{1}{64}x^4 + \&c..$$