Question 1. — Evaluate the following integerals.

a. $\displaystyle\int_{\frac{1}{3}\sqrt{3}}^{1} \frac{dx}{x\sqrt{4x^2-1}}$ b. $\displaystyle\int_0^{\frac{1}{4}\pi} \tan^2(x)\sec^4(x)\,dx$ c. $\displaystyle\int \frac{x^3+6x+8}{x^2(x^2+4)}\,dx$

d. $\displaystyle\int \frac{x-1}{x\sqrt{x-\log(x)}}\,dx$ e. $\displaystyle\int x^2\arcsin(x^3)\,dx$

f. $\displaystyle\int \frac{\sqrt{x^2+1}}{x^4}\,dx$ g. $\displaystyle\int \cos(x)\log(\cos x)\,dx$

Question 2. — Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0}\frac{x\sin(x)}{1-\cos(x)}$ b. $\displaystyle\lim_{x\to 1^+}\frac{\operatorname{arcsec}(x)}{\sqrt{x-1}}$ c. $\displaystyle\lim_{x\to 0}(x+e^{-x})^{1/x^2}$

Question 3. — Evaluate each improper integral or show that it diverges.

a. $\displaystyle\int_3^{\infty}\frac{dx}{x^2-2x+5}$ b. $\displaystyle\int_0^{1/e}\frac{dx}{x(\log x)^2}$

Question 4. — Find the area of the region enclosed by by $y = x^2 - 4x$ and $y = -x^2 + 6x - 8$.

Question 5. — Sketch the region $\mathscr{R}$ bounded by the graphs of $y = e^x$ and $y = -x$ from $x = 0$ to $x = 1$. Set up, but do not evaluate, an integral which represents the volume of the solid obtained by rotating $\mathscr{R}$ around: a. the line defined by $y = 4$; b. the line defined by $x = -2$.

Question 6. — Solve the differential equation

$$\sqrt{1-x^2}\,\frac{dy}{dx} = e^{3-y}$$

given that $y = 3$ when $x = 0$. Express y as a function of x.

Question 7. — A hydroelectric dam generates electricity by forcing water through turbines. This water carries sediment, some of which accumulateds behind the dam and some of which passes through the turbines. Assume sediment flows in from the river at a rate of 20 thousand tons per year, but each year $\frac{1}{10}$ of the accumulated sediment passes through the turbines. Find a formula for the amount of sedmient (in thousands of tons) accumulated after t years, assuming an initial accumulation of 0 tons. In the long run, how many tons of sediment will accumulate behind the dam?

Question 8. — Determine whether the sequence defined by

$$a_n = (-1)^n\frac{e^{2n}-4n}{e^{2n}+n^2}$$

converges or diverges. If the sequence converges find its limit; otherwise, explain why it diverges.

Question 9. — For the series $\displaystyle\sum_{n=1}^{\infty}\left[\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right]$,

a. give a simplified formula for the sum of its first n terms, and
b. compute the sum of the series or else explain why it is divergent.

Question 10. — Determine whether the series converges or diverges. Justify your answer.

a. $\displaystyle\sum_{n=1}^{\infty}\left[\frac{2}{\sqrt{n}}-\frac{1}{n^2}\right]$ b. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$

c. $\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}$ d. $\displaystyle\sum_{n=1}^{\infty}\frac{(4n^2+1)^n}{(\pi n)^{2n}}$

Question 11. — Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

a. $\displaystyle\sum_{n=0}^{\infty}\frac{\cos(n)}{3^n+1}$ b. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n-\log(n)}$

Question 12. — Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(x+3)^n}{n^3 5^n}.$$

Question 13. — Find the Taylor series of

$$\frac{1}{(4-3x)^2}$$

centred at 1. Express the series using summation notation, and write the first four terms of the series explictly.

Solution to question 1. — a. Integrating by inspection gives

$$\int_{\frac{1}{3}\sqrt{3}}^{1} \frac{2\,dx}{2x\sqrt{4x^2-1}} = \operatorname{arcsec}(2x)\Big|_{\frac{1}{3}\sqrt{3}}^{1} = \tfrac{1}{3}\pi - \tfrac{1}{6}\pi = \tfrac{1}{6}\pi.$$

b. If $t = \tan(x)$, then $dt = \sec^2(x)\,dx$ and $\sec^2(x) = t^2+1$; also, $t = 0$ if $x - 0$ and $t = 1$ if $x = \frac{1}{4}\pi$, so that

$$\int_0^{\frac{1}{4}\pi} \tan^2(x)\sec^4(x)\,dx = \int_0^1 t^2(t^2+1)\,dt = \left(\tfrac{1}{5}t^5 + \tfrac{1}{3}t^3\right)\Big|_0^1 = \tfrac{1}{5} + \tfrac{1}{3} = \tfrac{8}{15}.$$

c. The resolution into partial fractions of the integrand is

$$\frac{x^3+6x+8}{x^2(x^2+4)} = \frac{a}{x} + \frac{2}{x^2} + \frac{bx+c}{x^2+4},$$

in which the coefficient over x^2 is obtained by inspection (covering and evaluating). Clearing denominators gives $x^3+6x+8 = ax(x^2+4) + 2(x^2+4) + (bx+c)x^2$, and comparing the cubic, quadratic and linear coefficients then gives $a+b=1$, $c+2=0$ and $4a=6$. Thus, $a = \frac{3}{2}$, $b = -\frac{1}{2}$ and $c = -2$, so the integral in question is equal to

$$\int\left\{\frac{3}{2x} + \frac{2}{x^2} - \frac{x}{2(x^2+4)} - \frac{2}{x^2+4}\right\}dx = -\frac{2}{x} + \tfrac{1}{4}\log\!\left(\frac{x^6}{x^2+4}\right) - \arctan\!\left(\tfrac{1}{2}x\right).$$

d. If $y = \sqrt{x - \log(x)}$, then $y^2 = x - \log(x)$ and $2y\,dy = (1 - 1/x)\,dx$, so that

$$\int \frac{x-1}{x\sqrt{x-\log(x)}}\,dx = \int \frac{1-1/x}{\sqrt{x-\log(x)}}\,dx = \int \frac{2y}{y}\,dy = 2y = 2\sqrt{x-\log(x)}.$$

e. If $\vartheta = \arcsin(x^3)$ then $x^3 = \sin(\vartheta)$, $3x^2\,dx = \cos(\vartheta)\,d\vartheta$ and $\cos(\vartheta) = \sqrt{1-x^6}$, so that

$$\int x^2\arcsin(x^3)\,dx = \tfrac{1}{3}\int \vartheta\cos(\vartheta)\,d\vartheta = \tfrac{1}{3}\big(\vartheta\sin(\vartheta) + \cos(\vartheta)\big)$$
$$= \tfrac{1}{3}x^2\arcsin(x^2) + \tfrac{1}{3}\sqrt{1-x^6}.$$

f. If $y = x^{-1}\sqrt{x^2+1}$ then $y^2 = 1 + x^{-2}$ and $-y\,dy = dx/x^3$, so that

$$\int \frac{\sqrt{x^2+1}}{x^4}\,dx = \int \frac{\sqrt{x^2+1}}{x}\cdot\frac{dx}{x^3} = -\int y^2\,dy = -\tfrac{1}{3}y^3 = -\frac{(x^2+1)^{3/2}}{3x^2}.$$

g. Partial integration gives

$$\int \cos(x)\log(\cos x)\,dx = \sin(x)\log(\cos x) + \int \frac{\sin^2(x)}{\cos(x)}\,dx$$
$$= \sin(x)\log(\cos x) + \int(\sec(x) - \cos(x))\,dx$$
$$= \sin(x)\big(1 + \log(\cos x)\big) + \log(\sec(x) + \tan(x)).$$

Solution to question 2. — a. The Maclaurin expansions of the sine and cosine give

$$\lim_{x\to 0}\frac{x\sin(x)}{1-\cos(x)} = \lim_{x\to 0}\frac{x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \&c.}{\frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \&c.} = 2.$$

b. If $\vartheta = \operatorname{arcsec}(x)$, then $x - 1 = \sec(\vartheta) - 1 = (1-\cos(\vartheta)) \div \cos(\vartheta) = 2\sin^2\!\left(\frac{1}{2}\vartheta\right) \div \cos(\vartheta)$; thus,

$$\lim_{x\to 1^+}\frac{\operatorname{arcsec}(x)}{\sqrt{x-1}} = 2\lim_{\vartheta\to 0^+}\frac{\frac{1}{2}\vartheta\sqrt{\cos(\vartheta)}}{\sqrt{2}\sin\!\left(\frac{1}{2}\vartheta\right)} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

c. The limit $\lim_{t\to 0}(1+t)^{1/t} = e$, and the expansion $e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \&c.$, yield

$$\lim_{x\to 0}(x + e^{-x})^{1/x^2} = \lim_{x\to 0}\left(1 + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \&c.\right)^{\frac{1}{\frac{1}{2}x^2 - \frac{1}{6}x^3 + \&c.}\cdot\frac{\frac{1}{2}x^2 - \frac{1}{6}x^3 + \&c.}{x^2}}$$
$$= \sqrt{e}.$$

Solution to question 3. — a. Completing the square and integrating by inspection gives

$$\int_3^\infty \frac{dx}{x^2-2x+5} = \int_3^\infty \frac{dx}{(x-1)^2+4} = \lim_{t\to\infty}\tfrac{1}{2}\arctan\!\left(\tfrac{1}{2}(t-1)\right) - \tfrac{1}{8}\pi = \tfrac{1}{8}\pi.$$

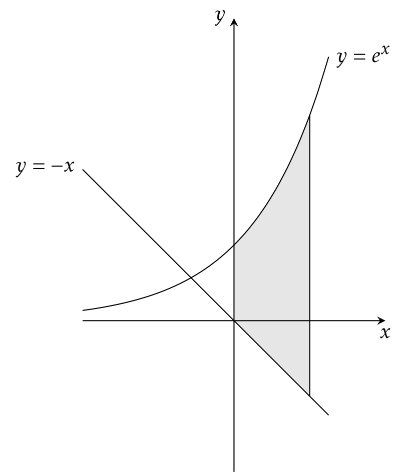
b. If $y = \log(x)$ then $dy = dx/x$, $y = -1$ if $x = 1/e$ and $y \to -\infty$ as $x \to 0^+$, so that

$$\int_0^{1/e}\frac{dx}{x(\log x)^2} = \int_{-\infty}^{-1}\frac{dy}{y^2} = 1 + \lim_{t\to-\infty}\frac{1}{t} = 1.$$

Solution to question 4. — The two curves meet where $x^2 - 4x = -x^2 + 6x - 8$, or equivalently, $0 = 2x^2 - 10x + 8 = 2(x-1)(x-4)$; *i.e.*, where $x = 1, 4$. If $1 < x < 4$ then $2(x-1)(x-4) < 0$, or $x^2 - 4x < -x^2 + 6x - 8$, so the area of the region enclosed by the curves is

$$\int_4^1 (2x^2 - 10x + 8)\,dx = \left(\tfrac{2}{3}x^3 - 5x^2 + 8x\right)\Big|_4^1 = \tfrac{2}{3}(-63) - 5(-15) + 8(-3) = 9.$$

Solution to question 5. — Below is a sketch in which the region $\mathscr{R}$ is shaded.



a. The solid obtained by revolving $\mathscr{R}$ about the line defined by $y = 4$ consists of annuli of inner radius $4 - e^x$ and outer radius $4 + x$, for $0 \leqslant x \leqslant 1$, so its volume is equal to

$$\pi\int_0^1\left\{(4+x)^2 - (4-e^x)^2\right\}dx.$$

b. The solid obtained by revolving $\mathscr{R}$ about the line defined by $x = -2$ consists of concentric cylindrical shells of radius $x + 2$ and height $e^x + x$, for $0 \leqslant x \leqslant 1$, so its volume is equal to

$$2\pi\int_0^1 (x+2)(e^x + x)\,dx.$$

Solution to question 6. — The differential equation is equivalent to

$$e^{y-3}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \qquad \text{and thus} \qquad e^{y-3} = 1 + \arcsin(x),$$

since $y = 3$ when $x = 0$. Solving for y then gives $y = 3 + \log(1 + \arcsin(x))$.

Solution to question 7. — If m denotes the mass (in thousands of tons) of sediment in the dam after t years, then

$$\frac{dm}{dt} = 20 - \tfrac{1}{10}m, \qquad \text{or} \qquad \frac{d}{dt}(m - 200) = -\tfrac{1}{10}(m - 200),$$

which is an exponential differential equation. Thus, $m - 200 = -200e^{-t/10}$, or $m = 200(1 - e^{-t/10})$, since $m = 0$ if $t = 0$. As $t \to \infty$, $m \to 200$, although there is certainly no realistic sense in which this constitutes “the long run.”

Solution to question 8. — By dominance ($0 < x^a e^{-bx} < (2a/b)^a e^{-bx/2}$ whenever $a, b, x > 0$)

$$\lim_{n\to\infty} \frac{e^{2n} - 4n}{e^{2n} + n^2} = 1.$$

Therefore, the sequence

$$\left\{(-1)^n \frac{e^{2n} - 4n}{e^{2n} + n^2}\right\}$$

oscillates and diverges (if n is very large and even its terms are very nearly 1, and if n is very large and odd its terms are very nearly -1).

Solution to question 9. — a. If $A_n = \cos(\pi/n)$ then the general term of the series is given by $a_n = A_n - A_{n+1}$, and the sum of the first n terms of the series is

$$a_1 + a_2 + \cdots + a_n = A_1 - A_{n+1} = -1 - \cos\left(\frac{\pi}{n+1}\right).$$

b. The sum of the series is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = -1 - \lim_{n\to\infty} \cos\left(\frac{\pi}{n+1}\right) = -2.$$

Solution to question 10. — a. If $n \geqslant 1$ then

$$a_n = \frac{2}{\sqrt{n}} - \frac{1}{n^2} \geqslant \frac{2}{\sqrt{n}} - \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}} > 0,$$

and $\sum n^{-1/2}$ is a divergent p-series ($p = \frac{1}{2} \leqslant 1$). Therefore, the comparison test implies that the series $\sum a_n$ is divergent.

b. Since $\lim_{t\to 0}(1+t)^{1/t} = e$, it follows that

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left\{\left(1 + \frac{1}{n}\right)^n\right\}^{-1} = e^{-1} \neq 0,$$

so the vanishing condition implies that the series $\sum\left(\frac{n}{n+1}\right)^n$ is divergent.

c. If $n \geqslant 1$ then

$$0 < a_n = \frac{(n!)^2}{(2n+1)!} = \frac{1}{n+1}\cdot\frac{2}{n+2}\cdot\frac{3}{n+3}\cdots\frac{n}{2n}\cdot\frac{1}{2n+1} < \left(\frac{1}{2}\right)^n,$$

and $\sum\left(\frac{1}{2}\right)^n$ is a convergent geometric series (its ratio, $\frac{1}{2}$, is positive and < 1). Therefore, the comparison test implies that the series $\sum a_n$ is convergent.

d. If $n > 1$ then

$$0 < a_n = \frac{(4n^2+1)^n}{(\pi n)^{2n}} < \frac{(5n^2)^n}{(\pi n)^{2n}} < \left(\frac{5}{9}\right)^n,$$

and the series $\sum\left(\frac{5}{9}\right)^n$ is a convergent geometric series (its ratio, $\frac{5}{9}$, is positive and < 1). Therefore, the comparison test implies that the series $\sum a_n$ is convergent.

Solution to question 11. — a. If $n \geqslant 0$ then $-1 \leqslant \cos(n) \leqslant 1$ and $3^n < 1 + 3^n$, so

$$-\left(\frac{1}{3}\right)^n < a_n = \frac{\cos(n)}{3^n + 1} < \left(\frac{1}{3}\right)^n.$$

Since $\sum\left(\frac{1}{3}\right)^n$ is a convergent geometric series (its ratio, $\frac{1}{3}$, is positive and < 1), the comparison test implies that $\sum a_n$ is absolutely convergent.

b. If $n \geqslant 2$ then $0 < n - \log(n) < n$ and thus

$$0 < \frac{1}{n} < a_n = \frac{1}{n - \log(n)},$$

so the comparison test implies that the series $\sum a_n$ diverges with the harmonic series. Thus, $\sum(-1)^n a_n$ is not absolutely convergent. On the other hand, $\lim a_n = 0$ since $n - \log(n) \to \infty$ as $n \to \infty$ by dominance ($0 < (\log x)^a x^{-b} < (2a/b)^a x^{-b/2}$ whenever $a, b > 0$ and $x > 1$). Also, if $n \geqslant 1$ then $n + 1 - \log(n+1) - (n - \log(n)) = 1 - \log(1 + 1/n) \geqslant 1 - \log(2) > 0$, so $a_n > a_{n+1} > 0$ for $n \geqslant 2$. Hence, the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent. Therefore, $\sum(-1)^n a_n$ is conditionally convergent.

Solution to question 12. — If $x \neq -3$ and

$$\alpha_n = \frac{(x+3)^n}{n^3 5^n}, \qquad \text{then} \qquad \lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \frac{1}{5}|x+3| \lim \frac{n^3}{(n+1)^3} = \frac{1}{5}|x+3|.$$

So the ratio test implies that the series $\sum \alpha_n$ is absolutely convergent if $|x+3| < 5$, *i.e.*, $-8 < x < 2$, and divergent if $x < -8$ or $x > 2$. If $x = 2$ then $\alpha_n = n^{-3}$ and if $x = -8$ then $\alpha_n = (-1)^n n^{-3}$. Since $\sum n^{-3}$ is a convergent p-series ($p = 3 > 1$), it follows that $\sum \alpha_n$ is absolutely convergent if $x = -8, 2$. Therefore, the interval of convergence of $\sum \alpha_n$ is $[-8, 2]$.

Solution to question 13. — Since $\frac{1}{(1-t)^2} = \frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{d}{dt}\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty}(n+1)t^n$, it follows that

$$\frac{1}{(4-3x)^2} = \frac{1}{(1-3(x-1))^2} = \sum_{n=0}^{\infty}(n+1)3^n(x-1)^n$$
$$= 1 + 6(x-1) + 27(x-1)^2 + 108(x-1)^3 + \&c.,$$

provided $-1 < 3(x-1) < 1$, or $\frac{2}{3} < x < \frac{4}{3}$.