Question 1. — Evaluate each of the following integrals.

a. $\displaystyle\int_0^1 \sqrt{2x-x^2}\,dx$ b. $\displaystyle\int \frac{e^{1/x}}{x^3}\,dx$ c. $\displaystyle\int \frac{dx}{(x+1)(x^2+1)}$

d. $\displaystyle\int \frac{\tan(x)}{\log(\cos(x))}\,dx$ e. $\displaystyle\int x\tan^2(x)\,dx$ f. $\displaystyle\int (\arcsin(x))^2\,dx$

Question 2. — Evaluate each of the following improper integrals.

a. $\displaystyle\int_{-\infty}^{\infty} \frac{\arctan(x)}{1+x^2}\,dx$ b. $\displaystyle\int_0^{\frac{1}{16}\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

Question 3. — Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 0}(1+3x)^{2\csc(x)}$ b. $\displaystyle\lim_{x\to 0}\frac{x\log(1+x)}{1-\cos(x)}$

Question 4. — Solve the ordinary differential equation

$$x+3y^2\sqrt{x^2+1}\,\frac{dy}{dx}=0$$

with the initial condition $y(0)=1$.

Question 5. — Find the area of thee region enclosed by the graphs of $y=\sin(x)$ and $y=\cos(x)$ between $x=0$ and $x=\pi$.

Question 6. — Sketch and shade the region $\mathscr{R}$ enclosed by the graphs of $y=x^3$ and $y=\sqrt{x}$ between $x=0$ and $x=\frac{1}{2}$. Set up, but do not evaluate, an integral which represents the volume of the solid obtained by rotating $\mathscr{R}$ about the line defined by:

a. $y=2$; b. $x=-2$.

Question 7. — Find the length of the curve $y=\log(1-x^2)$, $0\leqslant x\leqslant \frac{1}{2}$.

Question 8. — Determine the convergence or divergence of the sequence $\{(-1)^n n e^{-n}\}$. Justify your answer.

Question 9. — Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)}.$$

Question 10. — Determine whether each series is convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{1+e^{2n}}$ b. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+1}{3n+2}\right)^{n/2}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$

Question 11. — Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+1}}$ b. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{(2n)!}$

Question 12. — Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{2^n}{n}x^{2n}.$$

Question 13. — Find the Taylor series of $f(x)=e^{3x+1}$ centred at 2.

Question 14. — Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers which is decreasing and $\displaystyle\lim_{n\to\infty}a_n=2$. Prove that the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{a_n}{n}$ is conditionally convergent.

Solution to question 1. — a. Completing the square and letting $y = 1 - x$ gives

$$\int_0^1 \sqrt{2x - x^2}\,dx = \int_0^1 \sqrt{1 - (1-x)^2}\,dx = \int_0^1 \sqrt{1 - y^2}\,dy$$

Partial integration and absorption then gives

$$\int_0^1 \underbrace{1}_{\int} \cdot \underbrace{\sqrt{1-y^2}}_{D}\,dy = y\sqrt{1-y^2}\,\Big|_0^1 - \int_0^1 \frac{1 - y^2 + 1}{\sqrt{1-y^2}}\,dy$$

$$= 0 - \underbrace{\int_0^1 \sqrt{1-y^2}\,dy}_{\text{(absorb on the left)}} + \int_0^1 \frac{dy}{\sqrt{1-y^2}} = \tfrac{1}{2}\arcsin(y)\Big|_0^1 = \tfrac{1}{4}\pi.$$

b. If $y = 1/x$, then $-dy = dx/x^2$, then partial integration yields

$$\int \frac{e^{1/x}}{x^3}\,dx = \int \frac{1}{x}\cdot e^{1/x}\cdot\frac{dx}{x^2} = -\int \underbrace{y}_{D} \cdot \underbrace{e^y}_{\int}\,dy$$

$$= -ye^y + e^y = e^y(1-y) = e^{1/x}(1 - 1/x).$$

c. Resolving the integrand into partial fractions (the coefficient over $x+1$ is obtained by covering, and the coefficients over x^2+1 are obtained by comparing the quadratic and constant coefficients after clearing denominators) gives

$$\int \frac{dx}{(x+1)(x^2+1)} = \int \left\{ \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} \right\} dx$$

$$= \tfrac{1}{2}\log|x+1| - \tfrac{1}{4}\log(x^2+1) + \tfrac{1}{2}\arctan(x)$$

d. If $y = \log(\cos(x))$ then $dy = -\tan(x)\,dx$, so

$$\int \frac{\tan(x)}{\log(\cos(x))} = -\int \frac{dy}{y} = -\log|\log(\cos(x))|.$$

e. Writing $\tan^2(x) = \sec^2(x) - 1$ and then integrating by parts gives

$$\int x\tan^2(x)\,dx = \int \underbrace{x}_{D} \cdot \underbrace{(\sec^2(x) - 1)}_{\int}\,dx$$

$$= x(\tan(x) - x) - \int (\tan(x) - x)\,dx = x\tan(x) + \log|\cos(x)| - \tfrac{1}{2}x^2.$$

f. Partial integration gives

$$\int \underbrace{1}_{\int} \cdot \underbrace{(\arcsin(x))^2}_{D}\,dx = x(\arcsin(x))^2 - \int \underbrace{2\arcsin(x)}_{D}\,\underbrace{\frac{x}{\sqrt{1-x^2}}}_{\int}\cdot dx$$

$$= x(\arcsin(x))^2 + 2\arcsin(x)\sqrt{1-x^2} - 2\int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$$

$$= x(\arcsin(x))^2 + 2\arcsin(x)\sqrt{1-x^2} - 2x.$$

Solution to question 2. — a. If $y = \arctan(x)$, then $dy = \dfrac{dx}{1+x^2}$, $y \to -\frac{1}{2}\pi$ as $x \to -\infty$ and $y \to \frac{1}{2}\pi$ as $x \to \infty$, so that

$$\int_{-\infty}^{\infty} \frac{\arctan(x)}{1+x^2}\,dx = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} y\,dy = \tfrac{1}{2}y^2\Big|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = 0.$$

Notice that the integral becomes proper after the change of variable.

b. If $y = \sqrt{x}$, then $2\,dy = \dfrac{dx}{\sqrt{x}}$, $y \to 0^+$ as $x \to 0^+$ and $y = \frac{1}{4}\pi$ if $x = \frac{1}{16}\pi^2$; thus,

$$\int_0^{\frac{1}{16}\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = 2\int_0^{\frac{1}{4}\pi} \sin(y)\,dy = -2\cos(y)\Big|_0^{\frac{1}{4}\pi} = 2 - \sqrt{2}.$$

Notice that the integral becomes proper after the change of variable.

Solution to question 3. — a. Since $\lim\limits_{t\to 0}(1+t)^{1/t} = e$ (with $t = 3x$ in this case), it follows that

$$\lim_{x\to 0}(1+3x)^{2\csc(x)\cdot\frac{3x}{3x}} = \lim_{x\to 0}\Big((1+3x)^{1/(3x)}\Big)^{\frac{6x}{\sin(x)}} = e^6,$$

since $\lim\limits_{x\to 0}\dfrac{x}{\sin(x)} = 1$.

b. As $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c.$ and $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \&c.$, it follows that

$$\lim_{x\to 0}\frac{x\log(1+x)}{1-\cos(x)} = \lim_{x\to 0}\frac{x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \&c.}{\frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \&c.} = 2,$$

by inspecting the dominant terms.

Solution to question 4. — Separating the variables and integrating gives

$$\frac{x}{\sqrt{x^2+1}} + 3y^2\frac{dy}{dx} = 0, \qquad \text{and so} \qquad \sqrt{x^2+1} + y^3 = a,$$

for some constant a. The initial condition $y(0) = 1$ gives $a = 2$, so

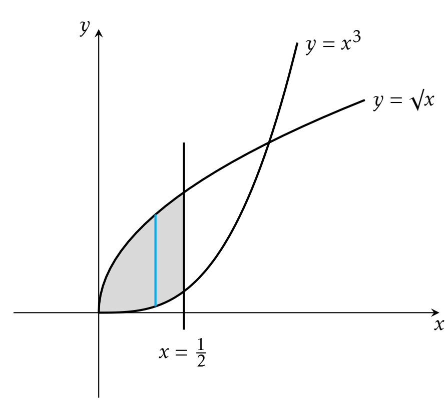
$$\sqrt{x^2+1} + y^3 = 2, \qquad \text{or} \qquad y = \sqrt[3]{2 - \sqrt{x^2+1}}.$$

Solution to question 5. — The curves meet where $\sin(x) = \cos(x)$, which is only where $x = \frac{1}{4}\pi$ in the interval $[0,\pi]$. If $0 < x < \frac{1}{4}\pi$ then $\cos(x) > \sin(x)$ and if $\frac{1}{4}\pi < x < \pi$ then $\cos(x) < \sin(x)$, so the area of the region is

$$\int_0^{\frac{1}{4}\pi} (\cos(x) - \sin(x))\,dx + \int_{\pi}^{\frac{1}{4}\pi} (\cos(x) - \sin(x))\,dx$$

$$= (\sin(x) + \cos(x))\Big|_0^{\frac{1}{4}\pi} + (\sin(x) + \cos(x))\Big|_{\pi}^{\frac{1}{4}\pi} = 2\sqrt{2}.$$

Solution to question 6. — Notice that $0 < x^3 < \sqrt{x}$ for $0 < x < \frac{1}{2}$. The region is sketched and shaded below, with a typical element (segment) drawn in cyan.



a. The solid obtained by revolving $\mathscr{R}$ about the line $y = 2$ consists of annuli of inner radius $2 - \sqrt{x}$ and outer radius $2 - x^3$, for $0 \leqslant x \leqslant \frac{1}{2}$, so its volume is equal to

$$\pi\int_0^{\frac{1}{2}} \Big\{(2 - x^3)^2 - (2 - \sqrt{x})^2\Big\}\,dx.$$

b. The solid obtained by revolving $\mathscr{R}$ about the line $x = -2$ consists of concentric cylindrical shells of radius $x+2$ and height $\sqrt{x} - x^3$, for $0 \leqslant x \leqslant \frac{1}{2}$, so its volume is equal to

$$2\pi \int_0^{\frac{1}{2}} (x+2)(\sqrt{x} - x^3)\,dx.$$

Solution to question 7. — First of all

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-2x}{1-x^2}\right)^2 = \frac{1 - 2x^2 + x^4 + 4x^2}{(1-x^2)^2} = \frac{1 + 2x^2 + x^4}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2,$$

so the length of the curve is

$$\int_0^{\frac{1}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\,dx = \int_0^{\frac{1}{2}} \left\{-1 - \frac{2}{x^2-1}\right\} dx = \left\{-x - \log\left|\frac{x-1}{x+1}\right|\right\}\Bigg|_0^{\frac{1}{2}} = -\tfrac{1}{2} - \log\left(\tfrac{1}{3}\right) = \log\left(3e^{-1/2}\right).$$

Solution to question 8. — Since

$$\lim_{n\to\infty} \left|(-1)^n n e^{-n}\right| = \lim_{n\to\infty} \frac{n}{e^n} = 0$$

by the dominance of exponential functions over power functions, it follows that the sequence $\{(-1)^n n e^{-n}\}$ converges to zero.

Solution to question 9. — Since

$$a_n = \frac{1}{(n+1)(n+3)} = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = A_n - A_{n+1},$$

where

$$A_n = \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right),$$

the sum of the first n terms of the series is

$$a_1 + a_2 + \cdots + a_n = A_1 - A_{n+1} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{n+2} + \frac{1}{+n+3}\right).$$

Plainly $\lim A_n = 0$, so the sum of the series is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = \lim_{n\to\infty}(A_1 - A_{n+1}) = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{3}\right) - 0 = \tfrac{5}{12}.$$

Solution to question 10. — a. If $a_n = \dfrac{e^n}{1 + e^{2n}}$ and $b_n = \dfrac{1}{e^n}$ then $a_n, b_n > 0$ for $n \geqslant 1$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left\{\frac{e^n}{1+e^{2n}} \cdot \frac{e^n}{1}\right\} = \lim_{n\to\infty} \frac{e^{2n}}{1 + e^{2n}} = 1 < \infty$$

by inspecting the dominant terms. Since $\sum b_n$ is a convergent geometric series ($r = \frac{1}{e}$, so $0 < r < 1$), the limit comparison test implies that the series $\sum a_n$ is also convergent.

b. If $a_n = \left(\dfrac{2n+1}{3n+2}\right)^{n/2}$ then $a_n > 0$ for $n \geqslant 1$ and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{2n+1}{3n+2}\right)^{1/2} = \left(\frac{2}{3}\right)^{1/2} < 1$$

(by inspecting the dominant terms), so the root test implies that the series $\sum a_n$ is convergent.

c. If $a_n = \dfrac{1}{n}\sin\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n^2}$ then $a_n, b_n > 0$ for $n \geqslant 1$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left\{\frac{1}{n}\sin\left(\frac{1}{n}\right) \cdot \frac{n^2}{1}\right\} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 < \infty,$$

since $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Since $\sum b_n$ is a convergent p-series ($p = 2$, so $p > 1$), the limit comparison test implies that the series $\sum a_n$ is convergent.

Solution to question 11. — a. Let

$$a_n = \frac{n}{\sqrt{n^3+1}} \qquad \text{and} \qquad b_n = \frac{1}{n^{1/2}},$$

then $a_n, b_n > 0$ for $n \geqslant 1$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} = \left\{\frac{n}{\sqrt{n^3+1}} \cdot \frac{n^{1/2}}{1}\right\} = 1,$$

by inspecting the dominant terms. Since $\lim(a_n/b_n) > 0$ and the series $\sum b_n$ is a divergent p-series ($p = \frac{1}{2} \leqslant 1$), the limit comparison test implies that the series $\sum a_n$ is divergent. Therefore, the series $\sum (-1)^n a_n$ is not absolutely convergent.

Next,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} = 0,$$

by inspecting the dominant terms, and

$$\frac{d}{dn}\left\{\frac{n}{\sqrt{n^3+1}}\right\} = \frac{1}{\sqrt{n^3+1}} - \frac{3n^2 \cdot n}{2(n^3+1)^{3/2}} = \frac{2 - n^3}{2(n^3+1)^{3/2}} < 0,$$

provided $n \geqslant 2$, so $a_n > a_{n+1}$ for $n \geqslant 2$. Therefore, the Leibniz test implies that the series $\sum (-1)^n a_n$ is convergent.

The series $\sum (-1)^n a_n$ is convergent but not absolutely convergent, so it is conditionally convergent.

b. If $a_n = (-1)^n \dfrac{e^n}{(2n)!}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left\{\frac{e^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{e^n}\right\} = \lim_{n\to\infty} \frac{e}{(2n+2)(2n+1)} = 0 < 1,$$

so the ratio test implies that the series $\sum a_n$ is absolutely convergent.

Solution to question 12. — If $\alpha_n = \dfrac{2^n}{n} x^{2n}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{n+1} x^{2n+2} \cdot \frac{2^n}{n} x^{2n}\right| = \lim_{n\to\infty} \frac{2nx^2}{n+1} = 2x^2,$$

by inspecting the dominant terms. So the ratio test implies that the series $\sum \alpha_n$ is absolutely convergent if $2x^2 < 1$ *i.e.*, if $-\frac{1}{2}\sqrt{2} < x < \frac{1}{2}\sqrt{2}$ and divergent if $x < -\frac{1}{2}\sqrt{2}$ or $x > \frac{1}{2}\sqrt{2}$. If $x = \pm\frac{1}{2}\sqrt{2}$ then

$$\alpha_n = \frac{2^n}{n} \cdot \left(\pm\tfrac{1}{2}\sqrt{2}\right)^{2n} = \frac{2^n}{n} \cdot \left(\tfrac{1}{2}\right)^n = \frac{1}{n},$$

so the series $\sum \alpha_n$ is the harmonic series, which is divergent. Therefore, the interval of convergence of the series $\sum \alpha_n$ is $\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$.

Solution to question 13. — Writing $3x + 1 = 7 + 3(x-2)$ and then using the Maclaurin series of the exponential function gives

$$e^{3x+1} = e^7 \cdot e^{3(x-2)} = e^7 \sum_{k=0}^{\infty} \frac{\left(3(x-2)\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{e^7 \cdot 3^k}{k!}(x-2)^k.$$

Solution to question 14. — Since $a_n > 0$ for $n \geqslant 1$,

$$\lim_{n\to\infty} \frac{a_n/n}{1/n} = \lim_{n\to\infty} a_n = 2 > 0$$

and the series $\sum 1/n$ is a divergent p-series ($p = 1$), the limit comparison test implies that the series $\sum a_n/n$ is divergent. Therefore, the series $\sum (-1)^n a_n/n$ is not absolutely convergent.

Next, $\displaystyle\lim_{n\to\infty} a_n/n = 0$ since $a_n \to 2$ and $n \to \infty$. It is given that $a_n > a_{n+1}$, for $n \geqslant 1$, so

$$\frac{a_n}{n} > \frac{a_{n+1}}{n} > \frac{a_{n+1}}{n+1},$$

i.e., the sequence $\{a_n/n\}$ is decreasing. Therefore, the Leibniz test implies that the series $\sum (-1)^n a_n$ is convergent.

The series $\sum (-1)^n a_n$ is convergent, but not absolutely convergent, so it is conditionally convergent.