Question 1. — Evaluate each of the following integrals.

a. $\displaystyle\int_0^{\frac{1}{4}\pi} \tan^4(\vartheta)\sec^4(\vartheta)\,d\vartheta$ b. $\displaystyle\int (\log(x))^2\,dx$ c. $\displaystyle\int \frac{\sec(x)\tan(x)}{\sqrt{9-\sec^2(x)}}\,dx$

d. $\displaystyle\int_1^2 \sqrt{3+2x-x^2}\,dx$ e. $\displaystyle\int \frac{x^4+3x^2+8}{x^4+4x^2}\,dx$

Question 2. — Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 1} \frac{\sin(\log(x))-x+1}{(x-1)^2}$ b. $\displaystyle\lim_{x\to-\infty} (1+e^x)^{e^{-x}}$

Question 3. — For each improper integral, either evaluate it or else explain why it is divergent.

a. $\displaystyle\int_0^{\frac{1}{2}\pi} \cot(x)\,dx$ b. $\displaystyle\int_0^{\infty} \frac{x}{e^x}\,dx$

Question 4. — Let $\mathscr{R}$ be the region enclosed by the x axis, and the graphs of $y=\sqrt{x}$ and $x=2-y^2$.

a. Sketch $\mathscr{R}$ and compute its area.
b. Find the volume of the solid obtained by rotating $\mathscr{R}$ about $x=2$.

Question 5. — Solve the initial value problem

$$\frac{1}{y}\frac{dy}{dx} - 2x\sqrt{y^2-1} = 0, \qquad y(0)=2.$$

Express y explicitly as a function of x.

Question 6. — According to Newton's law of cooling, the rate of change of the temperature of an object is proportional to the difference between its temperature and the ambient temperature. The temperature of a large room is kept at 20° C, and in that room a small object of temperature 100° C cools to 60° C after one minute.

a. Set up a differential equation for the temperature T of the object after t minutes, and solve the initial value problem.

b. How long will it take the object to cool to 30° C? Give a simplified exact answer.

Question 7. — Find the length of the curve $y=\log(\cos(x))$, $0 \leqslant x \leqslant \frac{1}{4}\pi$.

Question 8. — For each series, either find its sum or else show that it is divergent. Justify your answers.

a. $4-3+\dfrac{9}{4}-\dfrac{27}{16}+\dfrac{81}{64}-\cdots$ b. $\displaystyle\sum_{n=2}^{\infty}\left(e^{1/(n+1)}-e^{1/(n-1)}\right)$

Question 9. — Determine whether each series is convergent or divergent. Justify your conclusions.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan(n)}{1-e^{-n}}$ b. $\displaystyle\sum_{n=2}^{\infty}\frac{3-\cos^2(n)}{n-1}$

Question 10. — Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your conclusions.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{3^n(2n)!}$ b. $\displaystyle\sum_{n=1}^{\infty}(-2n)^n\big(\sin(1/n)\big)^n$ c. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n}{n^2+4}$

Question 11. — If a series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges conditionally and the limit $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, what can you say about this limit? Justify your answer.

Question 12. — Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^3 2^{n+1}}.$$

Question 13. — Find the Taylor series of $(1-2x)^{-1}$ centred at -1.

Solution to Question 1. — a. If $t = \tan(\vartheta)$ then $dt = \sec^2(\vartheta)\,d\vartheta$ and $\sec^2(\vartheta) = t^2+1$, so

$$\int_0^{\frac14\pi} \tan^4(\vartheta)\sec^4(\vartheta)\,d\vartheta = \int_0^1 t^4(t^2+1)\,dt = \int_0^1 (t^6+t^4)\,dt$$
$$= \left(\tfrac17 t^7 + \tfrac15 t^5\right)\Big|_0^1 = \tfrac17 + \tfrac15 = \tfrac{12}{35}.$$

b. Repeated partial integration gives

$$\int \underbrace{(\log(x))^2}_{D} \cdot \underbrace{dx}_{\int} = x(\log(x))^2 - 2\int \underbrace{\log(x)}_{D}\,\underbrace{dx}_{\int}$$
$$= x(\log(x))^2 - 2x\log(x) + \int 2\,dx = x(\log(x))^2 - 2x\log(x) + 2x.$$

c. If $s = \sec(x)$ then $ds = \sec(x)\tan(x)\,dx$, so

$$\int \frac{\sec(x)\tan(x)}{\sqrt{9-\sec^2(x)}}\,dx \int \frac{ds}{\sqrt{9-s^2}} = \arcsin\left(\tfrac13 s\right) = \arcsin\left(\tfrac13\sec(x)\right).$$

d. Since $3+2x-x^2 = 4-(x-1)^2 = 4-t^2$ with $t = x-1$, partial integration with absorption gives

$$\int_1^2 \sqrt{3+2x-x^2}\,dx = \int_0^1 \underbrace{\sqrt{4-t^2}}_{D} \cdot \underbrace{dt}_{\int} = t\sqrt{4-t^2}\,\Big|_0^1 + \int_0^1 \frac{t^2 - 4 + 4}{\sqrt{4-t^2}}\,dt$$
$$= \sqrt3 - \underbrace{\int_0^1 \sqrt{4-t^2}\,dt}_{\text{(absorb on the left)}} + 4\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \tfrac12\sqrt3 + 2\arcsin\left(\tfrac12 t\right)\Big|_0^1 = \tfrac12\sqrt3 + \tfrac13\pi.$$

e. Division and resolution into partial fractions yields

$$\frac{x^4+3x^2+8}{x^4+4x^2} = 1 + \frac{-x^2+8}{x^2(x^2+4)} = 1 + \frac{2}{x^2} - \frac{3}{x^2+4},$$

in which the coefficients are obtained by covering (taking x^2 as a variable). Thus,

$$\int \frac{x^4+3x^2+8}{x^4+4x^2}\,dx = x - \frac{2}{x} - \tfrac32\arctan\left(\tfrac12 x\right).$$

Solution to Question 2. — a. Expanding $\sin(\vartheta)$ about 0 and $\log(x)$ about 1 gives (via dominant terms)

$$\lim_{x\to1} \frac{\sin(\log(x)) - x + 1}{(x-1)^2}$$
$$= \lim_{x\to1} \frac{\left(-\frac12(x-1)^2 + \frac13(x-1)^3 - \cdots\right) - \frac16\left(x-1-\frac12(x-1)^2+\cdots\right)^3 + \cdots}{(x-1)^2}$$
$$= -\tfrac12.$$

Alternatively, two applications of l'Hôpital's rule with revision yields

$$\lim_{x\to1} \frac{\sin(\log(x)) - x + 1}{(x-1)^2} = \lim_{x\to1} \frac{\cos(\log(x))/x - 1}{2(x-1)}$$
$$= \lim_{x\to1} \frac{\cos(\log(x)) - x}{2(x^2-x)}$$
$$= \lim_{x\to1} \frac{-\sin(\log(x))/x - 1}{2(2x-1)}$$
$$= -\tfrac12$$

b. The limit $\lim_{t\to0}(1+t)^{1/t} = e$, with $t = e^x$ as $x \to -\infty$, gives

$$\lim_{x\to-\infty} (1+e^x)^{e^{-x}} = e.$$

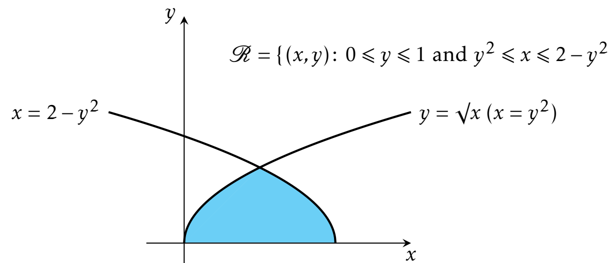
Solution to Question 3. — a. The integral diverges, since

$$\int_0^{\frac12\pi} \cot(x)\,dx = \lim_{x\to0^+} \int_x^{\frac12\pi} \cot(x)\,dx = \lim_{x\to0^+} \left(-\log|\sin(x)|\right) = \infty$$

b. Partial integration gives

$$\int_0^\infty \frac{x}{e^x}\,dx = \lim_{x\to\infty} \int_0^x \underbrace{x}_{D} \cdot \underbrace{e^{-x}}_{\int}\,dx = \lim_{x\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_0^x$$
$$= \lim_{x\to\infty} \left(1 - \frac{x+1}{e^x}\right) = 1.$$

Solution to Question 4. — a. The curves meet where $y > 0$, $x = y^2$ and $x = 2-y^2$; *i.e.*, where $x = y = 1$. Below is a sketch, with $\mathscr{R}$ shaded in cyan.



The area of the region $\mathscr{R}$ is thus

$$\int_0^1 \left(2 - y^2 - y^2\right)dy = \left(2y - \tfrac23 y^3\right)\Big|_0^1 = 2 - \tfrac23 = \tfrac43.$$

b. The solid obtained by revolving $\mathscr{R}$ about the line $x = 2$ consists of annuli of outer radius $2-y^2$ and inner radius $2-(2-y^2) = y^2$, for $0 \leqslant y \leqslant 1$, so its volume is equal to

$$\pi\int_0^1 \left((2-y^2)^2 - (y^2)^2\right)dy = \pi\int_0^1 (4-4y^2)\,dy = 4\pi\left(y - \tfrac13 y^3\right)\Big|_0^1 = \tfrac83\pi.$$

Solution to Question 5. — Revising the equation gives

$$\frac{1}{y\sqrt{y^2-1}}\frac{dy}{dx} = 2x, \qquad \text{or} \qquad \operatorname{arcsec}(y) = x^2 + \tfrac13\pi,$$

since $y(0) = 2$ and $\operatorname{arcsec}(2) = \frac13\pi$. Solving for y gives $y = \sec\left(x^2 + \frac13\pi\right)$.

Solution to Question 6. — a. If T is the temperature of the object t minutes after it has been in the room, then there is a positive real number $a = e^k$ such that

$$\frac{dT}{dt} = k(T-20), \quad \text{or} \quad \frac{d}{dt}(T-20) = k(T-20), \quad \text{so} \quad T - 20 = 80a^t,$$

since the equation is one of exponential decay and $T_0 = 100$. Since $T_1 = 60$, it follows that $a = \frac12$, so $T - 20 = 80\cdot 2^{-t}$.
b. The temperature of the object $T = 30$ when $10 = 80\cdot2^{-t}$, or $t = 3$. So the object cools to 30° C in three minutes.

Solution to Question 7. — If $y = \log(\cos(x))$ then

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+\tan^2(x)} = \sec(x),$$

since $0 \leqslant x \leqslant \frac14\pi$ (so $\sec(x) > 0$). Therefore, the length of the curve is

$$s = \int_0^{\frac14\pi} \sec(x)\,dx = \log(\tan(x) + \sec(x))\Big|_0^{\frac14\pi} = \log(1+\sqrt2).$$

Solution to Question 8. — a. Since no sequence is determined by any finite number of its terms, there is a series which begins as displayed and whose sum is 0. That is one perfectly valid answer.

Alternatively, the sequence $\left\{4\left(-\frac{3}{4}\right)^k\right\}_{k\geqslant 0}$ generates the displayed terms, and the sum of the corresponding geometric series is $\dfrac{4}{1-\left(-\frac{3}{4}\right)} = \frac{16}{7}$.

b. Since $a_n = e^{1/(n+1)} - e^{1/(n-1)} = A_{n+1} - A_n$, where $A_n = e^{1/n} + e^{1/(n-1)}$, the sum of the series is

$$\lim_{n\to\infty}(a_2 + a_3 + a_4 + \cdots + a_{n+1}) = \lim_{n\to\infty}(A_{n+2} - A_2) = 2 - A_2 = 2 - \sqrt{e} - e.$$

Solution to Question 9. — a. Since

$$\lim_{n\to\infty}\frac{\arctan(n)}{1-e^{-n}} = \tfrac{1}{2}\pi \neq 0,$$

the series in question diverges by the vanishing condition.

b. If $n \geqslant 2$ then $\cos^2(n) < 1$, so

$$a_n = \frac{3-\cos^2(n)}{n-1} > \frac{2}{n-1},$$

so the series $\sum a_n$ diverges by comparison with the harmonic series ($p = 1$).

Solution to Question 10. — a. If $a_n = \dfrac{(-1)^{n+1}n^2}{3^n(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left\{\left(\frac{n+1}{n}\right)^2\cdot\frac{1}{3}\cdot\frac{1}{(2n+2)(2n+1)}\right\} = 0 < 1$$

(via dominant terms), so $\sum a_n$ is absolutely convergent by the ratio test.

b. If $a_n = (-2n)^n\big(\sin(1/n)\big)^n$ then

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{2\sin(1/n)}{1/n} = 2 > 1,$$

since $\displaystyle\lim_{\vartheta\to 0}\frac{\sin(\vartheta)}{\vartheta} = 1$ (with $\vartheta = 1/n$), so $\sum a_n$ is divergent by the root test.

c. If $n > 2$ then

$$a_n = \frac{n}{n^2+4} > \frac{n}{n^2+n^2} = \frac{1}{2}\cdot\frac{1}{n},$$

so $\sum a_n$ diverges by comparison with the harmonic series ($p = 1$). [The limit comparison test could be used instead.] Therefore, $\sum(-1)^n a_n$ is not absolutely convergent. On the other hand, $\displaystyle\lim_{n\to\infty} a_n = 0$ (via dominant terms) and

$$\frac{d}{dn}\left(\frac{n}{n^2+4}\right) = \frac{4-n^2}{(n^2+4)^2} < 0, \quad \text{so} \quad a_n > a_{n+1}, \text{ provided } n > 2.$$

Therefore, the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent.

Since the series $\sum(-1)^n a_n$ is convergent but not absolutely convergent, it is conditionally convergent.

Solution to Question 11. — Let $\rho = \displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. If $\rho < 1$ then $\sum a_n$ is absolutely convergent and if $\rho > 1$ then $\sum a_n$ is divergent. Therefore, if $\sum a_n$ is conditionally convergent, ρ can only be $= 1$.

Solution to Question 12. — If $u_n = \dfrac{(x-2)^n}{n^3 2^{n+1}}$ and $x \neq 2$ then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \tfrac{1}{2}|x-2|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 = \tfrac{1}{2}|x-2|.$$

So the ratio test implies that $\sum u_n$ is absolutely convergent if $\frac{1}{2}|x-2| < 1$, *i.e.*, $0 < x < 4$, and $\sum u_n$ is divergent if $x < 0$ or $x > 4$. If $x = 0$ or 4 then $|u_n| = \frac{1}{2}n^{-3}$, so $\sum u_n$ is absolutely convergent (it is a multiple of a p-series with $p = 3 > 1$). Therefore, the radius of convergence of $\sum u_n$ is 2 and the interval of convergence of $\sum u_n$ is $[0, 4]$.

Solution to Question 13. — Expanding the geometric series gives

$$\frac{1}{1-2x} = \frac{1}{3-2(x+1)} = \frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}(x+1)} = \sum_{k=0}^{\infty}\frac{2^k}{3^{k+1}}(x+1)^k.$$

[Once could instead use the expansion of a binomial power.]