***Question 1***. — Evaluate the following integrals.

a. $\displaystyle\int_0^{\frac{1}{4}\pi} \tan^2(x)\sec^4(x)\,dx$ b. $\displaystyle\int e^x \sin(2x)\,dx$ c. $\displaystyle\int \frac{7x^2+2x-1}{x^2(x+1)}\,dx$

d. $\displaystyle\int_0^1 \frac{81x^5}{\sqrt{3x^3+1}}\,dx$ e. $\displaystyle\int_0^{\log(2)} \frac{x}{e^x}\,dx$ f. $\displaystyle\int \frac{\sqrt{9-4x^2}}{x}\,dx$

***Question 2***. — Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0} \frac{x\sin(x)}{1-\cos(x)}$ b. $\displaystyle\lim_{x\to 0}(x+e^{2x})^{1/x}$

***Question 3***. — Evaluate each improper integral or show that it diverges.

a. $\displaystyle\int_3^{\infty} \frac{dx}{x^2-2x+5}$ b. $\displaystyle\int_0^{1/e} \frac{dx}{x(\log(x))^2}$

***Question 4***. — Find the area of the region enclosed by the curves $y = x^2-4x$ and $y = -x^2+6x-8$.

***Question 5***. — Sketch the region $\mathscr{R}$ between the graphs of $y = e^x$ and $y = -x$ from $x = 0$ to $x = 1$. Set up, but do not evaluate, and integral which represents the volume of the solid obtained by revolving $\mathscr{R}$ about the line defined by:

a. $y = 4$; b. $x = -2$.

***Question 6***. — Solve the differential equation

$$\sqrt{1-x^2}\,\frac{dy}{dx} = \frac{1}{y},$$

given that $y = -\sqrt{\pi}$ when $x = \frac{1}{2}$. Express $y$ as a function of $x$.

***Question 7***. — The Bertalanffy equation is used by ecologists to model the growth of organisms over time. It is derived from the differential equation

$$\frac{dL}{dt} = k(L-M)$$

where $L$ is the length of the organism at time $t$, $M$ is the maximal length of the organism and $k$ is a rate constant. Suppose that upon first measurement a salmon is 20 cm long, and after one year that salmon is 50 cm long. If the maximal length is 80 cm, use this model to predict the length of the salmon two years after the first measurement.

***Question 8***. — Consider the sequence $\{a_n\}$ defined by $a_n = (-1)^n \dfrac{e^n-n}{e^n+n}$. Find the limit of this sequence or else explain why it is divergent.

***Question 9***. — Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2+(-3)^n}{5^n}$$

is convergent or divergent. If the series is convergent, find its sum.

***Question 10***. — Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Determine whether each item below is convergent or divergent. Justify your conclusions.

a. The sequence $\{a_n\}$ of terms of the series.
b. The sequence $\{a_1+a_2+a_3+\cdots+a_n\}$ of partial sums of the series.
c. The series $\sum_{n=1}^{\infty} e^{-a_n}$.

***Question 11***. — Determine whether each series converges or diverges. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{4n+5}$ b. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{(4n^2+1)^n}{(n\pi)^{2n}}$

***Question 12***. — Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n)}{3^n+1}$ b. $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^n}{n-2\sqrt{n}}$

***Question 13***. — Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^3 5^n}.$$

***Question 14***. — Find the Taylor series of $\dfrac{1}{(4-3x)^2}$ centred at 1. Write the first four non-zero terms explicitly and express the series in summation notation.

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***Solution to Question 1***. — a. If $y = \tan(x)$ then $dy = \sec^2(x)\,dx$ and $\sec^2(x) = y^2+1$, so

$$\int_0^{\frac14\pi} \tan^2(x)\sec^4(x)\,dx = \int_0^1 y^2(y^2+1)\,dy = \left(\tfrac15 y^5 + \tfrac13 y^3\right)\Big|_0^1 = \tfrac15+\tfrac13 = \tfrac{8}{15}.$$

b. Repeated partial integration with absorption yields

$$\int \underbrace{e^x}_{\int}\cdot\underbrace{\sin(2x)}_{D}\,dx = e^x\sin(2x) - 2e^x\cos(2x) - \underbrace{4\int e^x \sin(2x)\,dx}_{\text{(absorb on the left)}}$$
$$= \tfrac15 e^x\big(\sin(2x)-2\cos(2x)\big).$$

c. The resolution into partial fractions of the integrand is

$$\frac{7x^2+2x-1}{x^2(x+1)} = \frac{3}{x} - \frac{1}{x^2} + \frac{4}{x+1},$$

where the coefficients over $x^2$ and $x+1$ are obtained by covering and the coefficient over $x$ is obtained by comparing the quadratic coefficients. Thus,

$$\int \frac{7x^2+2x-1}{x^2(x+1)}\,dx = \log\left|x^3(x+1)^4\right| + \frac{1}{x}.$$

d. If $y = \sqrt{3x^3+1}$ then $y^2 = 3x^3+1$, $2y\,dy = 9x^2\,dx$, and $x^3 = \frac13(y^2-1)$, so $81x^5\,dx = 6y(y^2-1)\,dy$. Hence,

$$\int_0^1 \frac{81x^5}{\sqrt{3x^3+1}}\,dx = \int_1^2 \frac{6y(y^2-1)}{y}\,dy = (2y^3-6y)\Big|_1^2 = 8.$$

e. Partial integration gives

$$\int_0^{\log(2)} \underbrace{x}_{D}\cdot\underbrace{e^{-x}}_{\int}\,dx = (-xe^{-x}-e^{-x})\Big|_0^{\log(2)} = \tfrac12\log\left(\tfrac12 e\right).$$

f. If $y = \sqrt{9-4x^2}$ then $y^2 = 9-4x^2$ and $2y\,dy = -8x\,dx$, or $4x\,dx = -y\,dy$, and $4x^2 = 9-y^2$, so

$$\int \frac{\sqrt{9-4x^2}}{x}\,dx = \int \frac{\sqrt{9-4x^2}}{4x^2}\cdot 4x\,dx = \int \frac{y^2}{y^2-9}\,dy = \int\left\{1+\frac{9}{y^2-9}\right\}dy$$
$$= y + \tfrac32\log\left|\frac{y-3}{y+3}\right| = \sqrt{9-4x^2} + \tfrac32\log\left|\frac{\sqrt{9-4x^2}-3}{\sqrt{9-4x^2}+3}\right|$$
$$= \sqrt{9-4x^2} - 3\log\frac{3+\sqrt{9-4x^2}}{|x|}$$

***Solution to Question 2***. — a. The expansions of sine and cosine yield

$$\lim_{x\to0}\frac{x\sin(x)}{1-\cos(x)} = \lim_{x\to0}\frac{x^2-\frac16x^4+\frac{1}{120}x^6-\&c.}{\frac12x^2-\frac{1}{24}x^4+\frac{1}{720}x^6-\&c.} = 2$$

(via dominant terms).

Alternatively, two applications of l'Hôpital's rule gives

$$\lim_{x\to0}\frac{x\sin(x)}{1-\cos(x)} = \lim_{x\to0}\frac{\sin(x)+x\cos(x)}{\sin(x)} = \lim_{x\to0}\frac{2\cos(x)-x\sin(x)}{\cos(x)} = 2.$$

b. The limit $\lim_{t\to0}(1+t)^{1/t} = e$ gives (via dominant terms)

$$\lim_{x\to0}(x+e^{2x})^{1/x} = \lim_{x\to0}(x+e^{2x})^{\frac{1}{e^{2x}+x-1}\cdot\frac{e^{2x}+x-1}{x}} = e^3,$$

since $\dfrac{e^{2x}+x-1}{x} = \dfrac{3x+\frac12(2x)^2+\frac16(2x)^3+\&c.}{x}$ by the expansion of $e^{2x}$. Or the exponent can be obtained by one application of l'Hôpital's rule:

$$\lim_{x\to0}\frac{e^{2x}+x-1}{x} = \lim_{x\to0}\frac{2e^{2x}+1}{1} = 3.$$

***Solution to Question 3***. — a. Since $x^2-2x+5 = (x-1)^2+4$, the integral is

$$\lim_{x\to\infty}\int_3^x \frac{dx}{(x-1)^2+4} = \lim_{x\to\infty}\tfrac12\arctan\left(\tfrac12(x-1)\right)\Big|_3^x = \tfrac12\left(\tfrac12\pi-\tfrac14\pi\right) = \tfrac18\pi.$$

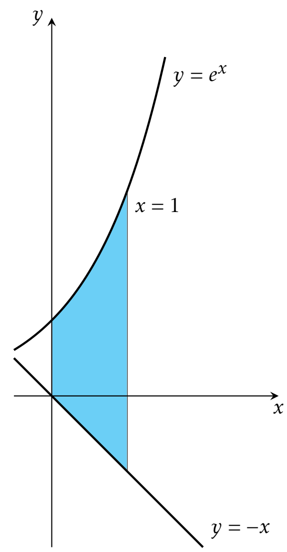
b. If $y = \log(x)$ then $dy = dx/x$, so that

$$\int_0^{1/e}\frac{dx}{x(\log(x))^2} = \lim_{y\to-\infty}\int_y^{-1}\frac{dy}{y^2} = \lim_{y\to-\infty}\left(-\frac1y\right)\Big|_y^{-1} = \lim_{y\to-\infty}\left(1-\frac1y\right) = 1.$$

***Solution to Question 4***. — The curves meet where $-x^2+6x-8 = x^2-4x$, or $0 = 2x^2-10x+8 = 2(x-1)(x-4)$. If $1<x<4$ the last difference is negative, so the area enclosed by the curves is (note the sign adjustment)

$$\int_4^1 (2x^2-10x+8)\,dx = \left(\tfrac23x^3-5x^2+8x\right)\Big|_4^1 = \tfrac23(-63)-5(-15)+8(-3) = 9.$$

***Solution to Question 5***. — Since $e^x > 0 \geqslant -x$ for $0\leqslant x\leqslant 1$, the curves do not intersect on the given interval. Below is a sketch in which the region $\mathscr{R}$ is shaded in cyan.




a. The solid obtained by revolving $\mathscr{R}$ about $y = 4$ consists of annuli of outer radius $4+x$ and inner radius $4-e^x$, for $0\leqslant x\leqslant 1$, so its volume is

$$\pi\int_0^1\left((4+x)^2-(4-e^x)^2\right)dx.$$

b. The solid obtained by revolving $\mathscr{R}$ about the $x = -2$ consists of concentric cylindrical shells of radius $x+2$ and height $e^x+x$, for $0\leqslant x\leqslant 1$, so its volume is

$$2\pi\int_0^1 (x+2)(e^x+x)\,dx.$$

***Solution to Question 6***. — Revising the equation gives

$$2y\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}, \qquad \text{or} \qquad y^2 = 2\arcsin(x)+\tfrac23\pi,$$

since $y = -\sqrt{\pi}$ when $x = \frac{1}{2}$ and $2\arcsin\left(\frac{1}{2}\right) = \frac{1}{3}\pi$. Solving for $y$ then gives $y = -\sqrt{2\arcsin(x) + \frac{2}{3}\pi}$. [ Note the sign. ]

***Solution to Question 7.*** — Since $M$ is a constant, the given equation is equivalent to

$$\frac{d}{dt}(L-M) = k(L-M), \qquad \text{and so} \qquad L-M = Aa^t$$

where $A$ is a constant and $a = e^k$. It follows that

$$a = \frac{L_2 - 80}{50-80} = \frac{50-80}{20-80} = \frac{1}{2}, \qquad \text{so} \qquad L_2 = 80 + \tfrac{1}{2}\cdot(-30) = 65.$$

Thus after two years the salmon is 65 cm long.

***Solution to Question 8.*** — Since $\lim_{n\to\infty} \frac{e^n - n}{e^n + n} = 1$ (via dominant terms), it follows that the sequence $\{a_n\}$ oscillates and diverges (its terms are nearly 1 if $n$ is large and even, and nearly $-1$ if $n$ is large and odd).

***Solution to Question 9.*** — The series in question is a linear combination of convergent geometric series:

$$\sum_{n=1}^{\infty} \frac{2+(-3)^n}{5^n} = \sum_{n=1}^{\infty} 2\left(\tfrac{1}{5}\right)^n + \sum_{n=1}^{\infty}\left(-\tfrac{3}{5}\right)^n = \frac{2}{5}\cdot\frac{1}{1-\frac{1}{5}} - \frac{3}{5}\cdot\frac{1}{1-\left(-\frac{3}{5}\right)}$$
$$= \tfrac{1}{2} - \tfrac{3}{8} = \tfrac{1}{8}.$$

***Solution to Question 10.*** — a. The sequence $\{a_n\}$ converges to 0 by the vanishing condition.
b. The sequence $\{a_1 + a_2 + a_3 + \cdots + a_n\}$ converges to the sum of the series.
c. Since $\lim_{n\to\infty} e^{-a_n} = 1 \neq 0$, $\sum e^{-a_n}$ is divergent by the vanishing condition.

***Solution to Question 11.*** — a. Since $\lim_{n\to\infty} \frac{2n+3}{4n+5} = \frac{1}{2} \neq 0$, the vanishing condition implies that the series in question is divergent.

b. Since

$$\lim_{k\to\infty}\left\{\frac{(k+1)!}{(k+1)^{k+1}}\cdot\frac{k^k}{k!}\right\} = \lim_{k\to\infty}\frac{k^k}{(k+1)^k} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{-k} = e^{-1} < 1,$$

the ratio test implies that the series in question is convergent.

c. Since

$$\lim_{n\to\infty} \sqrt[n]{\frac{(4n^2+1)^n}{(n\pi)^{2n}}}\lim_{n\to\infty}\frac{4n^2+1}{\pi^2 n^2} = \frac{4}{\pi^2} < 1,$$

the root test implies that the series in question is convergent.

***Solution to Question 12.*** — a. If $n \geqslant 0$ then

$$\left|\frac{\cos(n)}{3^n+1}\right| < \frac{1}{3^n+1} < \left(\tfrac{1}{3}\right)^n,$$

and $\sum\left(\frac{1}{3}\right)^n$ is a convergent geometric series ($r = \frac{1}{3}$, $|r| < 1$), the comparison test implies that the series in question is absolutely convergent.

b. If $n > 4$ then

$$a_n = \frac{1}{n - 2\sqrt{n}} > \frac{1}{n} > 0,$$

so $\sum a_n$ is divergent by comparison with the harmonic series ($p$-series, with $p = 1$). Thus, the series $\sum(-1)^n a_n$ is not absolutely convergent. But plainly $\lim_{n\to\infty} a_n = 0$, and if $n > 4$ (so that $\sqrt{n}$ and $\sqrt{n}-2$ are positive) then

$$a_n = \frac{1}{(\sqrt{n}-2)\sqrt{n}} > \frac{1}{(\sqrt{n+1}-2)\sqrt{n+1}} = a_{n+1},$$

so the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent.

Since the series $\sum(-1)^n a_n$ is convergent but not absolutely convergent, it is conditionally convergent.

***Solution to Question 13.*** — If $u_n = \frac{(x+3)^n}{n^3 5^n}$ and $x \neq -3$ then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \tfrac{1}{5}|x+3|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 = \tfrac{1}{5}|x+3|,$$

so the ratio test implies that the series $\sum u_n$ is absolutely convergent if $\frac{1}{5}|x+3| < 1$, *i.e.*, $-8 < x < 2$, and divergent if $x < -8$ or $x > 2$. If $x = -8, 2$ then $\sum |u_n| = \sum n^{-3}$ is a convergent $p$-series ($p = 3 > 1$), so the series $\sum u_n$ is absolutely convergent. Thus, the power series $\sum u_n$ has radius of convergence 5 and interval of convergence $[-8, 2]$.

***Solution to Question 14.*** — From the basic geometric series we have

$$\frac{1}{4-3x} = \frac{1}{1-3(x-1)} = \sum_{k=0}^{\infty} 3^k (x-1)^k,$$

so

$$\frac{1}{(4-3x)^2} = \frac{1}{3}\cdot\frac{d}{dx}\left(\frac{1}{4-3x}\right) = \sum_{k=1}^{\infty} k3^{k-1}(x-1)^{k-1}.$$

Alternatively, the expansion of a binomial power gives

$$\frac{1}{(4-3x)^2} = \left(1+(-3)(x-1)\right)^{-2}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{-2}{1}\cdot\frac{-3}{2}\cdot\frac{-4}{3}\cdots\frac{-(1+n)}{n}(-3)^n(x-1)^n$$
$$= \sum_{n=0}^{\infty}(1+n)3^n(x-1)^n.$$

[ Note that the first term is absorbed by the simplified pattern of terms. ] The two expansions are related by $k = n+1$, and the first four non-zero terms of the series are

$$1 + 6(x-1) + 27(x-1)^2 + 108(x-1)^3 + \cdots$$