CEGEP JOHN ABBOTT COLLEGE

LINEAR ALGEBRA

(MATHEMATICS 201-NYC-05/∞)

1. Solve, if possible:

   (a) $\begin{array}{rcl} x_1 - x_2 + 4x_3 + x_4 &=& 0 \\ 2x_1 - 2x_2 \qquad\quad - x_4 &=& 0 \end{array}$ (b) $\begin{array}{rcl} x + 2y &=& 3 \\ 2x + y &=& 2 \\ x + 5y &=& 1 \end{array}$

2. Is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2+t-s \\ 3+t+2s \\ 4-t \\ 1+s \end{pmatrix}$ a (general) solution to the following system?
   $$\begin{array}{rcl} x_1 \qquad + x_3 + x_4 &=& 7 \\ x_1 + x_2 \qquad - x_4 &=& 4 \end{array}$$

3. Determine all values of k (if any) so that the system $\begin{array}{rcl} x + 2y + kz &=& 1 \\ 2x + ky + 8 &=& 3 \end{array}$ has (a) no solution, (b) exactly one solution (c) infinitely many solutions.

4. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 0 & -5 \end{pmatrix}$. Show that $A^TA \neq AA^T$.

5. Consider the system of linear equations: $\begin{array}{rcl} x_1 + 2x_2 + 3x_3 &=& 1 \\ x_1 \qquad + 8x_3 &=& -6 \\ 2x_1 + 5x_2 + 3x_3 &=& 6 \end{array}$
   (a) Express the system as a matrix equation of the form $A\vec{x} = \vec{b}$
   (b) Find A^{-1}. (c) Use A^{-1} to solve the matrix equation $A\vec{x} = \vec{b}$ for $\vec{x}$.
   (d) Use Cramer's rule to solve for x_3 only.

6. (a) Find an LU factorization for $A = \begin{pmatrix} 2 & 1 & 2 \\ -4 & 1 & -3 \\ 1 & 10 & 7 \end{pmatrix}$
   (b) Use this LU factorization to solve $A\vec{x} = \begin{pmatrix} 12 & -26 & 14 \end{pmatrix}^T$.

7. $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
   Use the above to
   (a) write A^{-1} as a product of elementary matrices,
   (b) write A as a product of elementary matrices.

8. Let $\vec{x}_1,\ \vec{x}_2$ be different solutions to $A\vec{x} = \vec{0}$. Show that $\alpha\vec{x}_1 - \vec{x}_2$ is also a solution to $A\vec{x} = \vec{0}$.

9. If A and B are invertible $n \times n$ matrices, show that $(AB)^{-1} = B^{-1}A^{-1}$.

10. Evaluate $\det \begin{pmatrix} 5 & 0 & -1 & 0 \\ 0 & 0 & -3 & 2 \\ 1 & 4 & 0 & 4 \\ 1 & 2 & -1 & 0 \end{pmatrix}$.

11. Let A be a 3×3 matrix with $\det(A) = 4$. Find
    (a) $\det\left(A^TA^{-1}\right)$ (b) $\det\left((2A)^{-1}\right)$
    (c) the solution for ... here we have a problem: this part is missing from the electronic document.

12. With the aid of a vector diagram, find the point, R, that is one quarter of the way along the line segment from $P(2, 1, -2)$ to $Q(1, -2, 1)$.

13. Given
    $$\ell_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 3 \end{pmatrix} + s\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix} \text{ and } \ell_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix},$$
    $p_1\colon 2x - 3y + 4x + 9 = 0$, and $p_2\colon x - y + z - 1 = 0$.
    (a) Is ℓ_1 parallel to ℓ_2? Justify your answer.
    (b) Find the point of intersection of ℓ_1 and p_1.
    (c) Find the intersection, if any, of p_1 and p_2.
    (d) Find the acute angle between p_1 and p_2.

14. Given the points $A(2, -1, 3)$, $B(4, 0, 2)$ and $C(-2, 2, 1)$, find
    (a) the cartesian equation of the plane through the points A, B and C,
    (b) the area of $\triangle ABC$,
    (c) the distance between the point $P_0(2, 4, -1)$ and the plane in part (a),
    (d) the parametric equations for the line through A and B, and
    (e) the projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.

15. If $\vec{u},\ \vec{v},\ \vec{w} \in \mathbb{R}^3$, prove the following.
    (a) $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$
    (b) $(k\vec{u} + \vec{w}) \times \vec{u} = -(\vec{u} \times \vec{w})$

16. Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geqslant 0,\ y \geqslant 0 \right\}$.
    (a) Draw a picture of V.
    (b) Is V closed under addition?
    (c) Is V a subspace of $\mathbb{R}^2$?

17. The plane $x + 3y - 2z = 0$ is a two dimensional subspace of $\mathbb{R}^3$. Find a basis for this subspace.

18. Let $\mathscr{S} = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$
    Answer **true** or **false** (and explain your answer).
    (a) $\mathscr{S}$ is a basis for $\mathbb{R}^2$.
    (b) $\operatorname{Span}(\mathscr{S}) = \mathbb{R}^2$.
    (c) $\operatorname{Span}(\mathscr{S}) = \mathbb{R}^3$.
    (d) $\mathscr{S}$ is a linearly independent set of vectors.

19. (a) Give a condition on a and b so that $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.
    (b) Give a condition on a, b and c so that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$ is a linearly independent set.

20. Given $A = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 3 & -1 & 0 & 11 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 2 \\ -3 & 2 & 0 & -13 & 3 \\ 2 & 0 & 1 & 8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
    Find a basis for, and the dimension of
    (a) the row space of A
    (b) the column space of A
    (c) the null space of A
    (d) the null space of A^T