1. Solve the following system.

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & & & = & 4 \\ x_1 & & & + & x_3 & - & x_4 & = & 0 \\ & & 2x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}$$

2. Given the following system $A\vec{x} = \vec{b}$

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & a \\ & & 2x_2 & + & 3x_3 & = & b \\ x_1 & - & x_2 & - & 2x_3 & = & c \end{array}$$

   (a) Give a condition on a, b and c that must be satisfied in order for the system to be consistent.
   (b) Give a vector in $\mathbb{R}^3$ that cannot be written as a linear combination of the columns of A.

3. For what value(s) of k, if any, does the system

$$\begin{array}{rcrcl} x_1 & + & kx_2 & = & 2 \\ kx_1 & + & x_3 & = & 2 \end{array}$$

   have (a) a unique solution? (b) no solution? (c) infinitely many solutions?

4. Let $A = \begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$, and let $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.
   (a) Give the elementary matrix E_1 such that $E_1 A = B$.
   (b) Find two more elementary matrices E_2 and E_3 such that $E_3 E_2 E_1 A = C$.
   (c) Find elementary matrices F, G and H such that $A = FGHC$.

5. Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$.
   (a) Evaluate $BA - 2C$.
   (b) Find a matrix D, if possible, such that $C + D = C^T$.
   (c) Find a matrix E, if possible, such that $CE = C^T$.

6. Given that A, B, C and X are all invertible matrices, solve the equation $A(X + I)B = C$ for X.

7. Evaluate the determinant of the following matrix: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$

8. Given $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $\det(A) = -2$, find the following.

   (a) $\begin{vmatrix} 3a & g & d \\ 3b & h & e \\ 3c & i & f \end{vmatrix}$ (b) $\begin{vmatrix} a & b & c \\ d + 2a & e + 2b & f + 2c \\ g + 3a & h + 3b & i + 3c \end{vmatrix}$ (c) $\det(A^T A)$

9. Find the inverse of $\begin{pmatrix} 5 & 8 & 10 \\ 0 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}$.

10. Use matrix methods to balance the chemical equation

$$C_4H_{10} + O_2 \leftrightharpoons CO_2 + H_2O$$

11. (a) Find the point of intersection of the lines

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

   (b) Find the angle between the two lines in part (a).
   (c) Find an equation for the plane containing the two lines in part (a).

12. Given the points $A(0, 3, 1)$, $B(1, 2, 2)$, $C(3, 3, 6)$ and $O(0, 0, 0)$.
   (a) Find the projection of the vector $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.
   (b) Find the line through A and B.
   (c) Find a unit vector in the same direction as $\overrightarrow{AB}$.
   (d) Find the distance from C to the line through A and B.
   (e) Find the area of the triangle ABC.
   (f) Find the equation of the plane passing through A such that the vector $\overrightarrow{AB}$ is normal to the plane.
   (g) Find the volume of the parallelepiped having $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ as sides.

13. The plane $3x + y - 3z = 0$ is a subspace of $\mathbb{R}^3$.
   (a) Find a basis for this subspace.
   (b) Show that $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ belongs to this subspace.
   (c) Write $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$ as a linear combination of the vectors found in part (a).

14. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 4 \end{pmatrix}$, and $\vec{v}_4 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$.
   (a) Is $\vec{v}_4$ a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$? Justify your answer.
   (b) Are $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ linearly independent? Explain.
   (c) Are $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ linearly independent? Explain.

15. Let $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 = y^2 \right\}$
   (a) Is the zero vector in S?
   (b) Give a nonzero vector in S.
   (c) Is S closed under scalar multiplication? Justify your answer.
   (d) Is S closed under addition? Justify your answer.
   (e) Is S a subspace of $\mathbb{R}^2$? Explain.

16. Suppose you are given the following matrix A, and the reduced row echelon form of A.

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 2 & 4 & 3 & 1 \\ 1 & 2 & 4 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   Let $\vec{a}_1$ be the first column of A, let $\vec{a}_2$ be the second column of A, *etc.*
   (a) Is $\{\vec{a}_1, \vec{a}_2\}$ linearly independent? Justify.
   (b) Is $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ linearly independent? Justify.
   (c) Give a basis for the column space of A.
   (d) Express $\vec{a}_5$ as a linear combination of the basis vectors from part (c).
   (e) Give a basis for the nullspace of A.
   (f) Give a basis for the row space of A.
   (g) What is the rank of A.

17. Circle whichever statement is true for the given condition.
   (a) Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a set of vectors in $\mathbb{R}^3$, then
      (i) S must be linearly independent.
      (ii) S might be linearly independent.
      (iii) S cannot be linearly independent.
   (b) Suppose A and B are invertible $n \times n$ matrices, then
      (i) $A + B$ must be invertible.
      (ii) $A + B$ might be invertible.
      (iii) $A + B$ cannot be invertible.
   (c) Let A be an invertibel 4×4 matrix with $\det(A) = k$, then
      (i) $\det(2A^{-1}) = \frac{2}{k}$.
      (ii) $\det(2A^{-1}) = \frac{1}{16k}$.
      (iii) $\det(2A^{-1}) = \frac{1}{2k}$.
      (iv) $\det(2A^{-1}) = \frac{16}{k}$.
   (d) Let A be a 4×7 matrix of rank 4, then
      (i) The columns of A span $\mathbb{R}^4$.
      (ii) The rows of A span $\mathbb{R}^4$.
      (iii) The rows of A span $\mathbb{R}^7$.

18. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Show that if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is any vector in $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$, then $x + y + z = 0$.

19. Suppose A is a 3×3 matrix such that $(A + I)^{-1} = A - I$. Prove that $A^2 = 2I$.