Linear Algebra

(Mathematics 201-NYC-05/∞)

Part A. Attempt all questions

1. Solve the system

$$\begin{array}{rcrcrcrcl} & & x_2 & + & 3x_3 & - & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & 4x_3 & + & 3x_4 & = & 0 \\ 2x_1 & + & 3x_2 & + & 2x_3 & - & x_4 & = & 0 \\ -4x_1 & - & 3x_2 & + & 5x_3 & - & 4x_4 & = & 0 \end{array}$$

2. Find an LU factorization of $A = \begin{pmatrix} 2 & -3 & 1 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$

3. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -5 & 0 \\ 4 & 0 & 1 \end{pmatrix}$
   (a) Find A^{-1}
   (b) Write A^{-1} as a product of elementary matrices.
   (c) Write A as a product of elementary matrices.

4. Let $A = \begin{pmatrix} 2 & 4 & 2 \\ -1 & 0 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 3 \\ -4 & -3 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 6 & 8 \\ 2 & 3 \end{pmatrix}$
   (a) Evaluate $C(AB - I)$.
   (b) Find a matrix M such that $CM = A$.

5. Solve the matrix equation $\left(X^T + BC^T\right)^T = A$ for X.

6. Compute $\det(A)$, where $A = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 5 & 1 & 2 & 0 \\ 2 & 6 & 0 & -1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$.

7. If $\det(A) = 2$ and $\det(B) = 5$, calculate $\det\left(A^3B^{-1}A^TB^2\right)$

8. For which values of x, if any, is $\begin{pmatrix} 1 & x & x \\ 0 & 1 & x \\ -1 & x & 1 \end{pmatrix}$ invertible?

9. If $A^3 = 0$, show that $(I + A)^{-1} = I - A + A^2$.

10. Find a condition on the numbers a, b and c so that the following system is consistent.

$$\begin{array}{rcrcrcl} x & + & 3y & + & z & = & a \\ -x & - & 2y & + & z & = & b \\ 3x & + & 7y & - & z & = & c \end{array}$$

11. Solve the following system for z only using Cramer's Rule.

$$\begin{array}{rcrcrcr} -4x & - & y & + & 3z & = & 1 \\ 6x & + & 2y & - & z & = & 0 \\ 3x & + & 3y & + & 2z & = & -1 \end{array}$$

12. Given the points $A(-2, -4, 1)$ and $B(3, 4, -2)$.
    (a) Graph the points A and B and show the line through these two points.
    (b) Give the line through A and B.
    (c) Does the point $C(-7, -12, 4)$ lie on the line through A and B? (Justify your answer.)

13. Find the line which passes through the point $(4, 9, -5)$ and is
    (a) perpendicular to the plane $-3x + 8y + 2z = 15$.
    (b) parallel to the line $(2 + 3t, -2 - 4t, 5 + 2t)$.

14. Given the vectors $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$.
    (a) Determine the angle between $\vec{v}$ and $\vec{w}$ (in degrees, to two decimal place accuracy).
    (b) Determine the projection of $\vec{v}$ onto $\vec{w}$.
    (c) Determine the vector of length 5 units in the same direction as $\vec{v}$.
    (d) Determine a cartesian equation of the plane which passes through the point $(4, -3, -6)$ and is parallel to both $\vec{v}$ and $\vec{w}$.

15. Find the distance between the planes

$$5x - y + 2z = -3 \quad \text{and} \quad 10x - 2y + 4z = 8.$$

16. Determine the intersection, if any, of
    (a) the planes $6x + 2y - 5z = -10$ and $7x + y - 3z = -4$.
    (b) the line $(2 - 4t, 8 + t, -3 - 5t)$ and the plane $3x - 2y - 6z = -24$.

17. (a) Show that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$, and
    (b) express $\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ as a linear combination of these basis vectors.

18. Let $S = \{ \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) : a, b \text{ are integers} \}$.
    (a) Is S closed under addition? Explain.
    (b) Is S closed under scalar multiplication? Explain.
    (c) Is $\vec{0} \in S$?
    (d) Is S a subspace of $\mathbb{R}^2$?

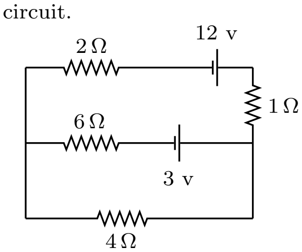
19. Let $A = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}$.
    (a) Find a basis for, and the dimension of:
    (i) $\mathrm{Col}(A)$, (ii) $\mathrm{Row}(A)$, (iii) $\mathrm{Nul}(A)$, (iv) $\mathrm{Nul}(A^T)$.
    (b) What is the rank of A?

20. The equation $2x_1 - 2x_2 + 3x_3 = 0$ defines a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

21. Let A be a 4×3 matrix with linearly independent columns. Which of the following are true? (*Briefly* justify your answer.)
    (a) $\mathrm{Nul}(A) = \{\vec{0}\}$.
    (b) The columns of A span $\mathbb{R}^4$.
    (c) The rows of A span $\mathbb{R}^3$.
    (d) If $A\vec{x} = \vec{b}$ is consistent, then it must have only one solution.

Part B. Attempt any 3 of the following 5 questions

1. Give an example of a nonzero 2×2 matrix that satisfies the condition, or explain why no such matrix exists.
   (a) $A^TA = -I$ (b) $A^T = 2A$
2. Determine the quadratic polynomial whose graph passes through the points $(1, 7)$, $(-1, 3)$ and $(2, 6)$.
3. Set up and reduce an augmented matrix to determine the currents in the given circuit.



4. Set up and reduce an augmented matrix to determine the coefficients necessary to balance

$$Br_2 + Na_2CO_3 \leftrightharpoons NaBr + NaBrO_2 + CO_2$$

5. A square matrix A is said to be *involutory* if $A^2 = I$.
   (a) Show that $\begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix}$ is involutory.
   (b) A student claims the product of two involutory matrices is also involutory, and gives the following "proof":
   Suppose $A^2 = I$ and $B^2 = I$; then
   $$(AB)^2 = A^2B^2 = I \cdot I = I$$
   What is wrong with this proof?