CEGEP JOHN ABBOTT COLLEGE

LINEAR ALGEBRA

(MATHEMATICS 201-NYC-05/∞)

PART I—SOLUTIONS OF LINEAR SYSTEMS AND APPLICATIONS

1. Given $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 4 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ find a matrix D such that $AB - 2D = C^2$.

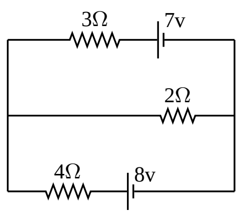
2. Solve the system $\left\{ \begin{array}{rcrcrcrcr} 2x_1 & + & 6x_2 & + & 9x_3 & - & 3x_4 & = & 8 \\ x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 3 \\ x_1 & + & 3x_2 & + & 5x_3 & - & x_4 & = & 5 \\ -x_1 & - & 3x_2 & - & x_3 & + & 5x_4 & = & 3 \end{array} \right.$

3. For which value(s) of k, if any, does the system $\left\{ \begin{array}{rcrcrcr} x & - & y & + & z & = & 0 \\ 2x & - & y & + & 4z & = & 0 \\ 3x & + & y & + & kz & = & 0 \end{array} \right.$ have
   (a) one solution? (b) no solution? (c) infinitely many solutions?

4. Set and solve a linear system to determine the integer coefficients which balance the chemical reaction equation
$$C_2H_6 + O_2 \longrightarrow CO_2 + H_2O$$

Do either question 5 or question 6. Do not do both.

5. Set up and solve an augmented matrix to determine the currents in the circuit:



6. Set up and solve an augmented matrix to determine the polynomial $p(x) = a_2x^2 + a_1x + a_0$ passing through the points $(1, 0)$, $(-2, -3)$ and $(2, 5)$.

PART II—VECTORS AND GEOMETRY

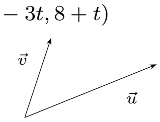
1. Given the points $A(1, -2, 3)$, $B(3, -4, 9)$ and $C(-4, 3, -10)$.
   (a) Find the line through A and B.
   (b) Show that the point C is *not* on the line through A and B.
   (c) Find the vector parallel but opposite in direction to $\overrightarrow{AB}$, and having a length of 10 units.
   (d) Find an equation for the plane perpendicular to $\overrightarrow{AB}$, and which contains the point C.

2. Given the vectors $\vec{u} = \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 5 \\ -3 \\ -4 \end{pmatrix}$.
   (a) Find the area of the parallelogram determined by by $\vec{u}$ and $\vec{v}$.
   (b) Find the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.
   (c) Find the angle between $\vec{u}$ and $\vec{v}$.
   (d) Find $\text{proj}_{\vec{w}}\vec{v}$ and $\text{perp}_{\vec{w}}\vec{v}$.

3. For the following pairs of planes, either find their intersection, or explain why there is no intersection and find the (shortest) distance between them.
   (a) $\left\{ \begin{array}{rcrcrcr} 3x & - & 5y & + & 2z & = & 2 \\ -5x & + & 7y & - & 4z & = & -6 \end{array} \right.$ (b) $\left\{ \begin{array}{rcrcrcr} 3x & - & 5y & + & 2z & = & 2 \\ -6x & + & 10y & - & 4z & = & 14 \end{array} \right.$

4. Find a cartesian equation for the plane containing the parallel lines
$$\ell_1 : (-3 - 4t, 1 + 6t, 6 - 2t) \qquad \ell_2 : (2 + 2t, -3 - 3t, 8 + t)$$

5. In the picture to the right, on the same diagram, draw the vectors $\text{proj}_{\vec{v}}\vec{u}$ and $\text{proj}_{\vec{u}}\vec{v}$.



6. Let ℓ be the line $(3 + 2t, 2 + at, -1 + t)$, $t \in \mathbb{R}$ and let ϖ be the plane $x + 2y + 4z = 4$.
   (a) For which values of a, if any, is ℓ parallel to ϖ?
   (b) For which values of a, if any, is ℓ perpendicular to ϖ?

PART III—INVERSES, TRANSPOSES AND LU-DECOMPOSITIONS

1. Given $A^{-1} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ -4 & -2 & -3 \end{pmatrix}$.
   (a) Find A.
   (b) Use A^{-1} to solve the system $A\vec{x} = \vec{b}$ for $\vec{x}$, where $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$. Find a matrix C such that $(AC + I)^T = A$.

3. Suppose that A and B are invertible $n \times n$ matrices. Which of the following are **true**, and which are **false**?
   (a) $(AB)^{-1} = A^{-1}B^{-1}$ (b) $(A + B)^T = A^T + B^T$
   (c) $(A + B)^{-1} = A^{-1} + B^{-1}$ (d) $\det(AB) = \det(BA)$

4. Given $A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -7 \\ -16 \\ 2 \end{pmatrix}$, use the LU method to solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

5. Given $A = \begin{pmatrix} 3 & 2 & 7 \\ 2 & 4 & 8 \\ 1 & 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 7 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 2 & 1/2 & -6 \\ 1 & 1 & -5 \\ -1 & -1/2 & 4 \end{pmatrix}$.
   (a) Find elementary matrices E_1 and E_2 such that $E_2E_1A = B$.
   (b) Use $E_2E_1A = B$ to find B^{-1}.

PART IV—DETERMINANTS AND VECTOR SPACES

1. Given $A = \begin{pmatrix} 1 & -1 & -5 & 6 & 51 \\ 0 & 1 & 3 & 0 & -4 \\ -2 & 2 & 10 & -11 & -95 \\ 3 & -3 & -15 & 17 & 146 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 & 5 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
   (a) Find a basis for the null space of A.
   (b) Find a basis for the columns space of A.
   (c) Identify the columns of A which are *not* in your basis for the column space of A, and express each such vector as a linear combination of your basis vectors.
   (d) Are the column vectors of A linear independent or dependent? Explain.
   (e) Find the dimension of the row space of A.

2. Given $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + 5y = 0 \right\}$.
   (a) Prove that S is a subspace of $\mathbb{R}^3$.
   (b) Describe in geometric terms the subspace S.
   (c) Find a basis for S.

3. Find k such that $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ k \end{pmatrix} \right\}$ is linearly dependent.

4. Determine whether the following sets of vectors are a basis for the given vector space V. Explain.
   (a) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}; V = \mathbb{R}^4$
   (b) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}; V = \mathbb{R}^2$
   (c) $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix} \right\}; V = \mathbb{R}^3$

5. Given $A\vec{x} = \vec{b}$ with $A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & -4 \\ -2 & 7 & 2 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ solve for x_3 only using Cramer's rule.

6. Given
$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad B = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$$
$$C = \begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ 5a - g & 5b - h & 5c - i \end{pmatrix} \qquad D = \begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ 3d & 3e & 3f \end{pmatrix}$$
and $\det A = -2$, evaluate
   (a) $\det B$ (b) $\text{rank}\, A$ (c) $\det C$ (d) $\det D$
   (e) $\det(3AB)$ (f) $\det(A^{-1}A^T$ (g) $\det\left((3A)^{-1}\right)$

7. Evaluate $\begin{vmatrix} 2 & 3 & 0 & 5 \\ 4 & 1 & 1 & 10 \\ -1 & 1 & 0 & -5 \\ 2 & 4 & -2 & 5 \end{vmatrix}$

8. Given $A^3 = A$, prove that $\det A = 0, \pm 1$.

PART I

1. $D = \begin{pmatrix} 4 & 2 \\ 4 & 3/2 \end{pmatrix}$
2. $(-3s+6t-5, s, 2-t, t)$
3. (a) $k \neq 11$, (b) none, (c) $k = 11$.
4. $2C_2H_6 + 7O_2 \longrightarrow 4CO_2 + 6H_2O$
5. The current in the upper loop is 1 A counterclockwise, and the current in the lower loop is 1 A clockwise. (The branch currents are: top: 1 A right to left, middle: 2 A left to right, bottom: 1 A right to left.)
6. $p(x) = x^2 + 2x - 3$.

PART II

1. (a) $(1+2t, -2-2t, 3+6t)$, (b) $\overrightarrow{AB} \not\parallel \overrightarrow{AC}$, (c) $\frac{10}{\sqrt{11}}(-1,1,-3)$, (d) $x-y+3z+37=0$.
2. (a) $10\sqrt{5}$ square units, (b) 58 cubic units, (c) $135.81°$, (d) $\vec{u}_{\vec{w}} = \begin{pmatrix} -3 \\ 9/5 \\ 12/5 \end{pmatrix}$; $\vec{u}_{\perp\vec{w}} = \begin{pmatrix} -1 \\ 1/5 \\ -7/5 \end{pmatrix}$
3. (a) $(4-3t, 2-t, 2t)$, (b) the shortest distance between the lines is $\frac{9}{38}\sqrt{38}$.
4. $2x - y - 7z + 49 = 0$
5. Get real.
6. (a) $a = -3$, (b) none.

PART III

1. (a) $A = \begin{pmatrix} -2 & -1/2 & -3/2 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$,
   (b) $\vec{x} = \begin{pmatrix} 10 \\ 9 \\ -17 \end{pmatrix}$.
2. $C = \begin{pmatrix} -1/2 & 1/2 \\ 7/2 & -7/2 \end{pmatrix}$
3. (a) False, (b) true, (c) false, (d) true.
4. $\vec{x} = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$
5. (a) $E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
   (b) $B^{-1} = \begin{pmatrix} -5 & 1/2 & 2 \\ -3 & 1 & 1 \\ 3 & -1/2 & -1 \end{pmatrix}$

PART IV

1. (a) $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 4 \\ 0 \\ -7 \\ 1 \end{pmatrix} \right\}$
   (b) $\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$
   (c) $\vec{a}_3 = -2\vec{a}_1 + 3\vec{a}_2$, $\vec{a}_5 = 5\vec{a}_1 - 4\vec{a}_2 + 7\vec{a}_4$
   (d) dependent—see part (c),
   (e) 3 (*i.e.*, the rank of the matrix)
2. (a) $S = \operatorname{Nul}(1\ 5\ 0)$, and is therefore a subspace.
   (b) S is a plane through the origin.
   (c) $\left\{ \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.
3. $k = 3$
4. (a) This is not a basis for $\mathbb{R}^4$; to be a basis for $\mathbb{R}^4$ requires 4 vectors. (b) This is not a basis for $\mathbb{R}^2$, because any set of more than two vectors in $\mathbb{R}^2$ is linearly dependent. (c) This is not a basis for $\mathbb{R}^3$ because the third vector is twice the first vector plus the second.
5. $x_3 = -16/107$
6. (a) -2, (b) 3, (c) 4, (d) 0, (e) 108, (f) 1, (g) $-1/54$.
7. $|A| = 45$
8. If $A^3 = A$ then $|A|^3 = |A|$, *i.e.*, $|A|$ is a solution of $0 = x^3 - x = x(x-1)(x+1)$, $\therefore$ $|A| = 0, \pm 1$.