CEGEP JOHN ABBOTT COLLEGE

LINEAR ALGEBRA

(MATHEMATICS 201-NYC-05/∞)

1. Solve the system: $\begin{cases} x + y - 2z + w = 4 \\ 2x - y + 2z + w = 5 \\ 3x + y - 2z + w = 10 \end{cases}$

2. Set up and solve a linear system to determine the integer coefficients which balance the chemical reaction equation:

$$I_2 + Na_2S_2O_3 \rightarrow NaI + Na_2S_4O_6$$

Define your variables and give the balanced equation.

3. Given the system: $\begin{cases} x + y = a \\ y + z = b \\ x - z = 0 \end{cases}$
Give conditions (if any) on a and b such that this system has
(a) one solution, (b) no solutions, (c) infinitely many solutions.

4. Let $B = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$,
   (a) find a matrix A such that $AB = B^T$.
   (b) find the general form of a 2×2 matrix C such that $(C-I)^T = I-C$.

5. Find an LU decomposition of $A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$.

6. Given $A = \begin{pmatrix} 0 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, find A^{-1}.

7. Suppose A is *any* square matrix such that $A^2 = I$.
Show that $(A+2I)^{-1} = -\frac{1}{3}(A-2I)$

8. let $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.
   (a) Find an elementary matrix, E_1, such that $E_1A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$.
   (b) Find an elementary matrix, E_2, such that $E_2E_1A = I$.
   (c) Use the result in (b) to express A as a product of elementary matrices.

9. Are the following true or false?
   (a) If A is invertible and $AB = AC$ then $B = C$.
   (b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$.
   (c) If E_1 and E_2 are elementary matrices, then $E_1 + E_2$ is elementary.
   (d) If L_1 and L_2 are lower triangular matrices, then $L_1 + L_2$ is lower triangular.
   (e) If A^{-1} and B^{-1} exist, then $(A+B)^{-1}$ exists.
   (f) If $A = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$, then $A^{-1} = A^T$.

10. Given the points $A(2,2,0)$, $B(-1,0,2)$ and $C(0,4,3)$, find
   (a) an equation for the line through A and C.
   (b) the coordinates of the point, R, that is one-quarter of the way from A to B.
   (c) a cartesian equation for the plane through A, B and C.
   (d) the distance from B to the line in part (a).

11. If $\vec{u} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$, then find
   (a) the angle between $\vec{u}$ and $\vec{w}$ (correct to two decimal places).
   (b) the projection of $\vec{u}$ onto $\vec{v}$.
   (c) the volume of the parallelepiped with sides $\vec{u}$, $\vec{v}$, $\vec{w}$.

12. Given:

$$\ell_1 : (9-5t, -1-t, 3+t) \qquad \ell_2 : (18+s, -2-4s, 6+s)$$

$$\varpi_1 : 2x - 3y + 4z + 9 = 0 \qquad \varpi_2 : x - y + z - 1 = 0$$

   (a) Is $\ell_1 \parallel \ell_2$? Justify.
   (b) Find the point if intersection, if any, of ℓ_1 and ϖ_1.
   (c) Find and describe the intersection of ϖ_1 and ϖ_2.

13. If $A^{-1} = \begin{pmatrix} -2 & 0 & 0 & -3 \\ 0 & -3 & 0 & 1 \\ 9 & 0 & 2 & 6 \\ 1 & 3 & 0 & 0 \end{pmatrix}$, then find
   (a) $\det(A^{-1})$ (b) $\det A$ (c) $\det(2A)$ (d) $\det\left(AA^T\right)$

14. Given $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & k-1 & 5 \\ 0 & -4 & -k \end{pmatrix}$.
   (a) Find the values of k for which A will not be invertible.
   (b) Find the values of k for which $A\vec{x} = \vec{b}$ will have a unique solution.

15. Using Cramer's rule, solve $\begin{cases} -3x + y - 2z = 2 \\ x + 2z = 4 \\ 2x + y + z = -1 \end{cases}$ for x only.

16. Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : z = 2x + 2y \right\}$.
   (a) Is S a subspace of $\mathbb{R}^3$? Justify.
   (b) find a basis for, and the dimension of, S.

17. Given that $S = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \right\}$:
   (a) Determine whether $\begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$ belongs to S.
   (b) Which of the following equations represent S? Justify.
      (i) $x - y - z = 0$
      (ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$
      (iii) $x + y - z = 0$

18. Suppose $\vec{v}_1 = \begin{pmatrix} 4 \\ 2 \\ -7 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -9 \end{pmatrix}$.
   (a) Is $\{\vec{v}_1, \vec{v}_2\}$ a basis for $\mathbb{R}^3$? Justify.
   (b) Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$? Justify.

19. If $\vec{u}$ and $\vec{v}$ are perpendicular vectors in $\mathbb{R}^n$, use properties of the dot product to prove that $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

20. In this problem you are given a matrix A and its reduced row echelon form. Let $\vec{a}_1$ represent the first column of A, $\vec{a}_2$ the second column of A, *etc.*

$$A = \begin{pmatrix} 1 & 0 & -1 & -1 & 6 \\ -2 & 1 & 4 & 4 & -17 \\ 0 & -2 & -4 & -3 & 6 \\ -1 & -3 & -5 & -5 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Find a basis for, and the dimension of, the null space of A.
   (b) Find a basis for, and the dimension of, the row space of A.
   (c) Find a basis for, and the dimension of, the column space of A.
   (d) Find the rank of A.
   (e) If possible, write $\vec{a}_5$ as a linear combination of the basis vectors for $\text{Col}(A)$.
   (f) Are $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_4$ linearly independent? Justify.

1. $(3, 1+2t, t, 0)$
2. $I_2 + 2Na_2S_2O_3 \to 2NaI + Na_2S_4O_6$
3. (a) Impossible. (b) $a \neq b$ (c) $a = b$
4. (a) $A = \frac{1}{5}\begin{pmatrix} 7 & -3 \\ -1 & 4 \end{pmatrix}$
   (b) $C = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}, t \in \mathbb{R}$.
5. $A = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$
6. $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 4 \\ -2 & 4 & -8 \\ 1 & 0 & 0 \end{pmatrix}$
7. $(A+2I)(-\frac{1}{3})(A-2I) = -\frac{1}{3}(-3I) = I$
8. (a) $E_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $E_2 = \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}$
   (c) $A = E_1^{-1}E_2^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$
9. (a) True, (b) false, (c) false, (d) true, (e) false, (f) true.
10. (a) $(2-2t, 2+2t, 3t)$ (b) $\left(\frac{5}{4}, \frac{3}{2}, \frac{1}{2}\right)$
    (c) $2x - y + 2z = 2$ (d) $\frac{15}{\sqrt{17}}$
11. (a) $136.04°$ (b) $\frac{15}{33}(1, 4, -4)$
    (c) 49 cubic units
12. (a) No, $\because$ the direction vector are not proportional. (b) $(-61, -15, 17)$
    (c) $(12+t, 11+2t, t)$; a line in $\mathbb{R}^3$.
13. (a) -6 (b) $-\frac{1}{6}$ (c) $-\frac{8}{3}$ (d) $\frac{1}{36}$
14. (a) $k = -4,\ 5$ (b) $k \neq -4,\ 5$
15. $x = -\frac{18}{7}$
16. (a) Yes, $\because S = \text{Nul}\begin{pmatrix} 2 & 2 & -1 \end{pmatrix}$.
    (b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$; $\dim(S) = 2$.
17. (a) Yes, $\because \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$
    (b) (i) Yes, $\because$ both vectors satisfy the equation, (ii) Yes, $\because$ this is a parametric representation of S, (iii) No, $\because$ *e.g.*, neither generating vector satisfies the equation.
18. (a) No, to generate $\mathbb{R}^3$ requires (at least) 3 vectors.
    (b) Yes, $\because \det\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = 20 \neq 0$.
19. $$\begin{aligned} \|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\ &= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \quad (\because \vec{u}\cdot\vec{v} = 0). \end{aligned}$$
20. (a) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}$, $\dim = 2$.
    (b) $\{\begin{pmatrix} 1 & 0 & -1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & -4 \end{pmatrix}\}$, $\dim = 3$. (c) $\{\vec{a}_1, \vec{a}_2, \vec{a}_4\}$, $\dim = 3$.
    (d) $\varrho(A) = 3$. (e) $\vec{a}_5 = 2\vec{a}_1 + 3\vec{a}_2 - 4\vec{a}_4$.
    (f) Yes, *e.g.*, $\because$ it is a basis for $\text{Col}(A)$.