SOLUTION OULINES (NYC, AUTUMN 2003)

1. $\left(\begin{array}{rrrrr|r} 1 & 4 & 0 & -1 & -2 & 3 \\ 2 & 8 & 3 & 4 & -13 & 0 \\ -3 & -12 & -2 & -1 & 12 & -5 \end{array}\right) \sim \left(\begin{array}{rrrrr|r} 1 & 4 & 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 2 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$;

   $(3 - 4r + s + 2t,\ r,\ -2 - 2s + 3t,\ s,\ t)$

2. $\left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 5 \\ 1 & 2 & 4 & 8 & 17 \\ 1 & -2 & 4 & -8 & 5 \end{array}\right)$; $\quad [p(t) = -1 - 5t + 3t^2 + 2t^3]$

3. $\left(\begin{array}{rrr|r} 3 & 2 & 0 & 5 \\ 1 & 4 & 3 & 5 \\ 0 & 1 & 3 & 22 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 10 \end{array}\right)$;

   Left loop, 7 A counterclockwise, middle loop, 8 A counterclockwise, right loop 10 A counterclockwise.
   $[I_1 = 7$ A, $I_2 = -1$ A, $I_3 = 2$ A, $I_4 = 10$ A$]$

4. $\left(\begin{array}{rrr|rr} 2 & 3 & -8 & 5 & 2 \\ 4 & 1 & -2 & 3 & 3 \\ -3 & -2 & 5 & -4 & -4 \end{array}\right) \sim \left(\begin{array}{rrr|rr} 1 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{7}{10} \\ 0 & 1 & -\frac{14}{5} & \frac{7}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right)$

   (a) $\left(\frac{2}{5} - t,\ \frac{7}{5} + 14t,\ 5t\right)$; this is a line in $\mathbb{R}^3$ through the point $\left(\frac{2}{5}, \frac{7}{5}, 0\right)$ with direction $(-1, 14, 5)^T$.
   (b) There is no intersection; pairwise, the planes intersect in three pairwise parallel non-coplanar lines (or: the intersection of any two planes is a line parallel to, but not lying in, the third plane).

5. $\vec{y} = \frac{1}{2}(\vec{u} + \vec{v})$, $\vec{x} = \frac{1}{2}(\vec{v} - \vec{u})$.

6. (a) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$; $6x - y - 3z = 12$
   (b) $D(5, 9, 3)$; $(5 - 6t,\ 9 + t,\ 3 + 3t)$
   (c) $A = \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{46} \quad (= \|\overrightarrow{AB} \times \overrightarrow{AD}\|)$

7. (a) $(2, 1, 2)$
   (b) $\left(2 \mp \frac{3}{7}\sqrt{14},\ 1 \pm \frac{2}{7}\sqrt{14},\ 2 \pm \frac{1}{7}\sqrt{14}\right)$
   (c) No.
   (d) $\vartheta = \arccos\left(\frac{2}{7}\sqrt{6}\right) \approx 45.58°$.

8. (a) The lines are parallel; $\frac{3}{11}\sqrt{814}$; $x - 3y - 8z = -27$.
   (b) The lines are skew; $\frac{9}{2}\sqrt{2}$.

9. (a) $\dfrac{1}{2}\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$
   (b) $\dfrac{1}{2}\begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}$

10. $AA^T = \begin{pmatrix} a^2 + b^2 & a(b+c) \\ a(b+c) & a^2 + c^2 \end{pmatrix}$, $A^TA = \begin{pmatrix} a^2 + c^2 & a(b+c) \\ a(b+c) & a^2 + b^2 \end{pmatrix}$.
    If $AA^T = A^TA$ then $b^2 = c^2$, so $b = -c$ ($\because b \neq c$).
    $\therefore AA^T = (a^2 + c^2)I$, and the conclusion follows.

11. (a) $B^{-1} = -\frac{1}{2}A^{-1} = -\dfrac{1}{2}\begin{pmatrix} -6 & -6 & 3 \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{pmatrix}$
    (b) $\vec{x} = \begin{pmatrix} 12 \\ 2 \\ -8 \end{pmatrix}$
    (c) $A = \dfrac{1}{6}\begin{pmatrix} -6 & -3 & -12 \\ 2 & 0 & 6 \\ -6 & -6 & -12 \end{pmatrix}$

12. (a) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 1/2 & 1 & 0 \\ 1 & 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 & 1 \\ 0 & 2 & -4 & -2 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}$
    (b) $\det A = 60$
    (c) $\det\left(3A^{-1}\right) = \frac{27}{20}$
    (d) $-\frac{8}{5}$

13. (a) If P is idempotent and invertible, then $P = P^{-1}P^2 = P^{-1}P = I$.
    (b) $(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$.

14. $E = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$, $F = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{smallmatrix}\right)$, $G = \left(\begin{smallmatrix} -1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $H = \left(\begin{smallmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

15. (a) Yes, if *e.g.*, $\vec{u} \neq \vec{0}$ and $\{\vec{u},\ \vec{v}\}$ is linearly dependent.
    (b) Yes, if $\{\vec{u},\ \vec{v}\}$ is linearly independent.
    (c) No, $\because \dim \operatorname{Span} S \leqslant 2$.

16. (a) The set of vectors in $\mathbb{R}^4$ which are orthogonal to $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ is a subspace of $\mathbb{R}^4$ which contains V (since it contains the given vectors spanning V).
    (b) No, because the zero vector does not belong to W.

17. (a) The set of vectors in $\mathbb{R}^2$ with positive entries.
    (b) The set of vectors in $\mathbb{R}^2$, the product of whose entries is nonnegative.

18. (a), (b), (c) and (e) are equal to S; (d) is not.

19. (a) The third pivot obtained by Gaussian elimination is $1 - 2k$, so the columns of A generate $\mathbb{R}^3$ if, and only if, $k \neq \frac{1}{2}$.
    (b) None, because the fourth column is twice the first (more generally, because four vectors in $\mathbb{R}^3$ cannot be linearly indpendent).
    (c) $k = \frac{1}{2}$. (See the answer to (a), or notice that if $k = \frac{1}{2}$, then the third row is twice the second row.)

20. (a) Yes. If $B = \begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{pmatrix}$ then $AB = \begin{pmatrix} A\vec{b}_1 & \cdots & A\vec{b}_n \end{pmatrix}$.
    (b) If $B\vec{x} = \vec{0}$ then $(AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$.
    (c) If $\operatorname{rk} B = \operatorname{rk} AB = \varrho$, then $\dim \operatorname{Nul} B = \dim \operatorname{Nul} AB = n - \varrho$ by part (a). Since $\operatorname{Nul} B$ is a subset of $\operatorname{Nul} AB$ by part (b), it follows that $\operatorname{Nul} B = \operatorname{Nul} AB$. (Pedantically, any basis of $\operatorname{Nul} B$ can be extended to a basis of $\operatorname{Nul} AB$, but since the dimensions of these subspaces are equal, no vectors are added in the process.)

21. (a) The first, second and fifth columns of A form a basis for $\operatorname{Col} A$, whose dimension is 3.
    (b) $\vec{a}_3 = \vec{a}_1 - 2\vec{a}_2$
    (c) The first three rows of R form a basis for $\operatorname{Row} A$, whose dimension is 3.
    (d) $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\operatorname{Nul} A$, whose dimension is 2.