LINEAR ALGEBRA
(MATHEMATICS 201-NYC-05)

1. Given the linear system $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 + 3x_5 &= 3 \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 1 \\ x_1 + 3x_2 + x_3 + 2x_4 + 2x_5 &= 0 \\ 3x_2 - x_3 + x_4 + 3x_5 &= 1 \end{aligned}$$

a. Solve this system.
b. What is the solution set of $A\mathbf{x} = \mathbf{0}$? (This should require no additional computation.)
c. Is the solution set of $A\mathbf{x} = \mathbf{b}$ a subspace of $\mathbb{R}^5$?

2. The augmented matrix $\begin{pmatrix} A & \mathbf{b} \end{pmatrix}$ of a linear system is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & x \\ 2 & 3 & 4 & 5 & y \\ 1 & 1 & 1 & 1 & z \end{pmatrix}$$

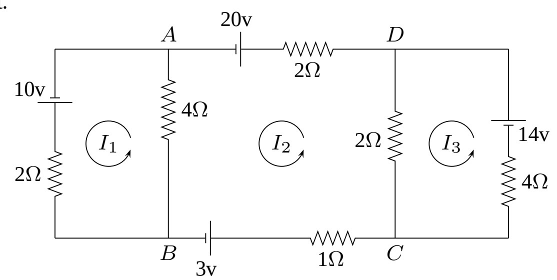
a. Write this augmented matrix in echelon form.
b. For which values of x, y, and z (if any) does this system have
i. no solution? ii. a unique solution? iii. infinitely many solutions?
c. For what value of k is $\begin{pmatrix} 72 \\ 31 \\ k \end{pmatrix}$ in Col A?

3. a. Let

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

i. Find A^{-1}, and ii. use A^{-1} to solve $XA = B$ for X.

b. By solving an appropriate linear system, find the loop currents in the following circuit.



4. You are given the following matrix A, and its reduced echelon form R.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 2 \\ 6 & 12 & 18 & 24 & 31 & 2 & 11 \\ 0 & 3 & 6 & 9 & 12 & 0 & -6 \\ 7 & 17 & 27 & 37 & 51 & 4 & 4 \\ 1 & 5 & 9 & 13 & 20 & 3 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a. Give a basis for $\operatorname{Col} A$.
b. Give a basis for $\operatorname{Nul} A$.
c. What is the rank of A?
d. Is the column space of A the same as the column space of R?
e. Is the null space of A the same as the null space of R?

5. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

a. What is the LU factorization of A?
b. What is $\det A$?
c. What is $\det(A - I)$?
d. Based only on the above results, what is $\det(A^2) - \det A$?
e. Based only on the above results, what is $\det(A^2 - A)$?

6. a. Give **specific numerical examples** of each of the following.
i. Two non-zero orthogonal vectors in $\mathbb{R}^4$.
ii. A 2×2 matrix A such that $\operatorname{Col} A = \operatorname{Row} A$.
iii. Two invertible matrices, A and B of the same size such that their sum $A + B$ is not invertible.
iv. A 2×2 matrix A such that $A^2 = I$ but $A \neq \pm I$.
v. Two non-zero 2×2 matrices whose product is the zero 2×2 matrix.

b. In the following sentences, fill in the the missing word. The missing word is **must**, **might** or **cannot**.
i. If A is an $n \times n$ matrix and $\det(A) = \det(2A)$ then A ___ be invertible.
ii. If $A\mathbf{x} = \mathbf{0}$ is consistent then A ___ be invertible.
iii. If $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent and $\{\mathbf{v}, \mathbf{w}\}$ is linearly dependent, then $\{\mathbf{u}, \mathbf{w}\}$ ___ be linearly dependent.

c. In the following statement, fill in the missing number. The missing number is 3, 5, 8 or 13.
If A is a 5×8 matrix then $\operatorname{rank} A + \dim \operatorname{Nul} A$ is ___.

d. In the following statement, fill in the missing word.
If $\operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$ then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly ___.

7. Let $\mathscr{H}$ be the vector space of all 3×3 symmetric matrices which have all zeros on the main diagonal.

a. Is the zero 3×3 matrix in $\mathscr{H}$?
b. Give an example of a non-zero 3×3 matrix X in $\mathscr{H}$.
Give an example of a 3×3 matrix Y that is *not* in $\mathscr{H}$.
c. Find a basis for $\mathscr{H}$.
d. What is the dimension of $\mathscr{H}$?

8. Let $\mathscr{P}$ be the parallelepiped formed by the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix},$$

and let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -5 & 2 \\ -3 & -5 & 0 \end{pmatrix}.$$

a. What is the volume of $\mathscr{P}$?
b. Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ a basis for $\mathbb{R}^3$? Explain your answer.
c. What is the volume of $T(\mathscr{P})$?
d. Is $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ a basis for $\mathbb{R}^3$? Explain your answer.

9. Let ϖ be the two dimensional subspace of $\mathbb{R}^3$ defined by

$$x - 3y + 2z = 0,$$

and let ℓ be the line defined by

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

a. Find a basis for ϖ.
b. Find the intersection of the plane ϖ and the line ℓ.
c. Write an equation of the line parallel to ℓ that passes through the origin.

10. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$.
a. Explain why there are scalars α and β such that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \alpha\mathbf{v} + \beta\mathbf{w}$.
In the rest of this problem, you will find expressions for α and β.
b. First, consider the simpler problem of finding scalars γ and δ such that

$$\mathbf{v} \times (\mathbf{v} \times \mathbf{w}) = \gamma\mathbf{v} + \delta\mathbf{w}. \qquad (*)$$

i. Take the dot product of the equation $(*)$ with $\mathbf{v}$ and simplify.
ii. Take the dot product of the equation $(*)$ with $\mathbf{w}$ and simplify.
iii. You now have two linear equations in γ and δ. Solve for γ and δ.
c. Take the dot product of $\mathbf{v}$ with the equation $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \alpha\mathbf{v} + \beta\mathbf{w}$ and simplify (use part b).
d. Give expressions for α and β in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.