1. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & 0 & 4 & 3 \\ 1 & 2 & 4 & 1 \\ 1 & 2 & 8 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

a. Put the matrix $\begin{pmatrix} A & \mathbf{b} \end{pmatrix}$ into echelon form.

b. Give a condition on b_1, b_2, b_3 such that the linear system is consistent.

c. Give a basis for the column space of A.

d. Give, if possible, a specific vector in $\mathbb{R}^3$ that is not in the column space of A. If this is not possible, then explain why it is not possible.

e. $\mathfrak{True}$ or $\mathfrak{false}$: The intersection of $\operatorname{col} A$ with any two dimensional subspace of $\mathbb{R}^3$ must contain infinitely many vectors. Justify your answer.

f. Give a basis for the null space of A.

2. Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -5 \\ -1 \\ -4 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{pmatrix} 3 \\ 13 \\ 4 \\ 9 \end{pmatrix}.$$

a. If possible, express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ using specific numerical coefficients. If this is not possible, explain why not.

b. What is the dimension of $\operatorname{Span} S$?

c. Give a basis for $\operatorname{Span} S$ consisting of unit vectors.

d. Give, if possible, a basis for $\operatorname{Span} S$ in which each vector has only negative entries. If this is not possible, then explain why it is not possible.

e. $\mathfrak{True}$ or $\mathfrak{false}$: Every two vector subset of S is linearly independent.

f. $\mathfrak{True}$ or $\mathfrak{false}$: Every three vector subset of S is linearly independent.

g. $\mathfrak{True}$ or $\mathfrak{false}$: Every two vector subset of $\operatorname{Span} S$ is linearly independent.

3. a. Given that X, Y and Z are invertible $n \times n$ matrices such that $XYZ = -I_n$, express Y^{-1} in terms of X and Z.

b. If M is an invertible $n \times n$ matrix, what is the inverse of $\begin{pmatrix} I_n & M \\ -I_n & 0 \end{pmatrix}$?

c. Find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}.$$

d. Use the inverse of A that you just computed to find the interpolating quadratic polynomial $y = c_0 + c_1 x + c_2 x^2$ for the data points $(1,3)$, $(2,5)$, $(3,6)$.

4. Let

$$A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}.$$

a. Evaluate $(A + A^{-1})^{-1}$.

b. For what value(s) of k is $A + kA^{-1}$ singular?

c. Evaluate A^2.

d. Find a matrix B such that $B + A = 2BA$.

e. Find an elementary matrix E such that EA is lower triangular.

5. a. Evaluate and simplify $\det A$, where

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \alpha & \beta \\ 1 & \alpha & 0 & \gamma \\ 1 & \beta & \gamma & 0 \end{pmatrix}.$$

b. Use Cramer's rule to solve the following system for x_2 only.

$$\begin{aligned} 2x_1 + 3x_2 - 3x_3 &= 0 \\ 2x_1 + 4x_2 + 3x_3 &= 5 \\ 3x_1 - 2x_2 - 2x_3 &= 0. \end{aligned}$$

6. a. Give, if possible, specific examples of each of the following. When this is not possible, explain why it is not possible.

i. An upper triangular 3×3 matrix with determinant 2.

ii. A non-invertible 2×2 matrix with determinant 3.

iii. A 2×4 matrix of rank 1.

iv. A subset $\mathbb{R}^2$ that is closed under addition but not closed under scalar multiplication.

b. Fill in the missing numerical value in each case.

i. If A is a 3×3 matrix and $\det A = 7$, then $\det(10A) =$ ________.

ii. If A is a 20×50 matrix of rank 11 then the dimension of $\operatorname{nul} A$ is ________.

iii. If $\begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$, then a is ________.

iv. If $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$, then the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is ________.

7. Let ϖ_1 be the plane with equation $2x - y + z = 3$ and let ϖ_2 be the plane with equation $x + y + 2z = 1$.

a. Find an equation for the line of intersection of ϖ_1 and ϖ_2.

b. For which values of a and b does $\begin{pmatrix} 11 \\ a \\ b \end{pmatrix}$ lie on this line of intersection?

c. Find an equation of the plane, ϖ_3, which is parallel to ϖ_2 and passes through the point $P(1,2,3)$.

d. Find the distance from ϖ_2 to ϖ_3.

8. In $\mathbb{R}^3$ the vectors $\mathbf{e}_1$, $2\mathbf{e}_2$ and $3\mathbf{e}_3$ are the vertices of a triangle $\mathscr{T}$.

a. What is the area of the triangle $\mathscr{T}$?

b. Find an implicit (cartesian) equation of the plane containing the triangle $\mathscr{T}$.

c. Find the distance from the vertex $3\mathbf{e}_3$ to the line through $\mathbf{e}_1$ and $2\mathbf{e}_2$.

d. The triangle $\mathscr{T}$ is rotated about the line through $\mathbf{e}_1$ and $2\mathbf{e}_2$ so that its image lies in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. What is the (acute) angle of this rotation?

9. Given vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, with $\mathbf{v} \neq \mathbf{0}$, let $\mathscr{C}$ denote the circle with diameter the line segment joining $\mathbf{u}$ and $\mathbf{v}$.

a. Express the centre of the circle $\mathscr{C}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

b. Express the radius of the circle $\mathscr{C}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

c. Let d be the distance from the origin to the center of the circle $\mathscr{C}$, and let r be the radius of the circle. Show that $d^2 - r^2 = \mathbf{u} \cdot \mathbf{v}$.

d. Does $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ lie on $\mathscr{C}$? Justify your answer.

10. Two matrices A and B are said to *commute* if $AB = BA$, and they are said to *anti-commute* if $AB = -BA$. Let M be an $n \times n$ matrix, and let $\mathscr{S}$ denote the set of all $n \times n$ matrices which anti-commute with M.

a. Show that $\mathscr{S}$ is a subspace of the linear space of all $n \times n$ matrices.

b. If $n = 2$ and

$$M = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

then give a basis for the set of matrices that anti-commute with M, and give a specific non-zero 2×2 matrix that anti-commutes with M.

c. Show that if A and B anti-commute, then A^2 and B^2 commute.

d. Suppose that A and B anti-commute. Find and simplify $(A + B)^2$.

e. Suppose that A and B anti-commute. Find and simplify $(A + B)^4$.

1. a. The row operations $R_1 \leftrightarrow R_2$, $R_3 \leftarrow R_3 - R_1$ and $R_3 \leftarrow R_3 - R_2$, give

$$\begin{pmatrix} A & \mathbf{b} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 & 1 & b_2 \\ 0 & 0 & 4 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{pmatrix}$$

b. The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if, and only if, $b_1 + b_2 = b_3$.

c. The first and third columns of A are the pivot columns of A, so

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is a basis for the column space of A.

d. The vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

does not belong to the column space of A, since it fails to satisfy the condition found in part (b).

e. This statement is $\mathfrak{true}$. The intersection of $\operatorname{col} A$ and any two dimensional subspace of $\mathbb{R}^3$ is infinite, because it is the solution set of a homogeneous system of two linear equations in three unknowns.

f. The list of vectors

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ -3 \\ 4 \end{pmatrix} \right\}$$

is a basis for the null space of A.

2. First, observe that

$$V = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & 2 & -5 & 13 \\ 1 & 1 & -1 & 4 \\ 2 & 1 & -4 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a. From the reduced echelon form of V, it is manifest that $\mathbf{v}_4 = 5\mathbf{v}_1 - \mathbf{v}_2$.

b. Since $\mathscr{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\operatorname{Span} S$, the dimension of $\operatorname{Span} S$ is two.

c. The basis obtained by normalizing the elements of $\mathscr{V}$, namely

$$\hat{\mathscr{V}} = \{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2\} = \left\{ \tfrac{1}{15}\sqrt{15} \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \tfrac{1}{10}\sqrt{10} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\},$$

is a basis for $\operatorname{Span} S$ that consists of unit vectors.

d. The basis consisting of the additive inverses of the elements of $\mathscr{V}$, namely

$$-\mathscr{V} = \{-\mathbf{v}_1, -\mathbf{v}_2\} = \left\{ \begin{pmatrix} -1 \\ -3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ -1 \\ -1 \end{pmatrix} \right\},$$

is a basis for $\operatorname{Span} S$ in which each vector has only negative entries.

e. Since no vector in S is a multiple of any other vector in S, it is $\mathfrak{true}$ that *every two vector subset of S is linearly independent*.

f. The statement in question is $\mathfrak{false}$. In fact, from the reduced echelon form of V one can conclude that *every three vector subset of S* (or even $\operatorname{Span} S$) *is linearly dependent*. (Note specifically that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly dependent by the solution to part a.)

g. The statement in question is $\mathfrak{false}$. For example, $\{\mathbf{0}, \mathbf{v}_1\}$ is a two vector subset of $\operatorname{Span} S$ that is linearly dependent.

3. a. We have $YZ = -X^{-1}$, or $Y(-ZX) = I_n$, so $Y^{-1} = -ZX$, by (the first corollary to) the invertible matrix theorem.

b. If

$$\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_n & M \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} A - B & AM \\ C - D & CM \end{pmatrix},$$

then $A = 0$ since $AM = 0$ and M is invertible. From this and $I_n = A - B$, it follows that $B = -I_n$. Next, $I_n = CM$ implies that $C = M^{-1}$ (since C and M are $n \times n$ matrices). Finally, $0 = C - D$ implies that $D = M^{-1}$. Therefore,

$$\begin{pmatrix} I_n & M \\ -I_n & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -I_n \\ M^{-1} & M^{-1} \end{pmatrix}.$$

c. By any method, one finds that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}.$$

d. Observe that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix},$$

and hence

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 7/2 \\ -1/2 \end{pmatrix};$$

therefore, the required polynomial is $y = \frac{7}{2}x - \frac{1}{2}x^2$.

4. a. We have

$$A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}, \quad A + A^{-1} = \begin{pmatrix} 1/2 & 3 \\ 3/2 & -1 \end{pmatrix},$$

and hence

$$(A + A^{-1})^{-1} = \begin{pmatrix} 1/5 & 3/5 \\ 3/10 & -1/10 \end{pmatrix}.$$

b. As

$$\begin{aligned} \det(A + kA^{-1}) &= \begin{vmatrix} \frac{1}{2}k & k+2 \\ \frac{1}{2}k + 1 & -1 \end{vmatrix} \\ &= -\tfrac{1}{2}k^2 - \tfrac{5}{2}k - 2 \\ &= -\tfrac{1}{2}(k+4)(k+1), \end{aligned}$$

$A + kA^{-1}$ is singular if, and only if, $k = -4$ or $k = -1$.

c.

$$A^2 = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}.$$

d. $B + A = 2BA$ is equivalent to $A = 2BA - B = B(2A - I)$, or

$$\begin{aligned} B = A(2A - I)^{-1} &= \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ 2/5 & 1/5 \end{pmatrix} \\ &= \begin{pmatrix} 4/5 & 2/5 \\ 1/5 & 3/5 \end{pmatrix}. \end{aligned}$$

e. Let E be the 2×2 elementary matrix that corresponds to the elementary row operation of adding twice row two to row one; then

$$EA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

is lower triangular.

5. a. Subtract column two from columns three and four, expand along row one, then subtract row one from rows two and three and expand along column one, and finally evaluate the resulting 2×2 determinant directly.

$$\begin{aligned} \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \alpha & \beta \\ 1 & \alpha & 0 & \gamma \\ 1 & \beta & \gamma & 0 \end{vmatrix} &= -\begin{vmatrix} 1 & \alpha & \beta \\ 1 & -\alpha & \gamma - \alpha \\ 1 & \gamma - \beta & -\beta \end{vmatrix} = -\begin{vmatrix} -2\alpha & \gamma - \alpha - \beta \\ \gamma - \alpha - \beta & -2\beta \end{vmatrix} \\ &= (\alpha + \beta - \gamma)^2 - 4\alpha\beta. \end{aligned}$$

b. Cramer's rule gives

$$x_2 = \begin{vmatrix} 2 & 0 & -3 \\ 2 & 5 & 3 \\ 3 & 0 & -2 \end{vmatrix} \div \begin{vmatrix} 2 & 3 & -3 \\ 2 & 4 & 3 \\ 3 & -2 & -2 \end{vmatrix} = \tfrac{25}{83}.$$

6. a. i. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is an upper triangular matrix with determinant equal to two.

ii. There is no such matrix, since any square matrix with non-zero determinant is invertible.

iii. $\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}$ is a 2×4 matrix of rank one.

iv. $Q = \{\, \mathbf{x} \in \mathbb{R}^2 \colon x_1 \geqslant 0 \text{ and } x_2 \geqslant 0 \,\}$ is a subset of $\mathbb{R}^2$ that is closed under addition, but not closed under scalar multiplication.

b. i. If A is a 3×3 matrix and $\det A = 7$, then $\det(10A) = 10^3 \cdot 7 = 7000$.

ii. If A is a 20×50 matrix of rank 11 then the $\dim \operatorname{nul} A$ is $50 - 11 = 39$.

iii. If $\begin{pmatrix}3\\-1\\a\end{pmatrix}$ is orthogonal to $\begin{pmatrix}4\\5\\3\end{pmatrix}$, then $7 + 3a = 0$, and so $a = -\frac{7}{3}$.

iv. The parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is $|\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}| = 5$.

7. a. One equation for the intersection of ϖ_1 and ϖ_2 is $(2x - y - z - 3)^2 + (x + y + 2z - 1)^2 = 0$. A parametric vector equation for this intersection (found by solving the system of their cartesian equations) is

$$\mathbf{x} = \begin{pmatrix}4/3\\-1/3\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\-1\end{pmatrix}.$$

b. For the given point to lie on the line, we must have $\frac{4}{3} + t = 11$, or $t = \frac{29}{3}$, and so $a = -\frac{1}{3} + \frac{29}{3} = \frac{28}{3}$ and $b = -\frac{29}{3}$.

c. An equation for ϖ_3 is $x + y + 2z = 9$ (the left hand side is given by that of ϖ_2 and the right hand side is given by the condition that $(1, 2, 3)$ lie on ϖ_3).

d. The distance between ϖ_2 and ϖ_3 is equal to the length of the projection of

$$\mathbf{v} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\3\end{pmatrix} \quad \text{onto their common normal} \quad \mathbf{n} = \begin{pmatrix}1\\1\\2\end{pmatrix},$$

which is given by $\dfrac{|\mathbf{v} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{4}{3}\sqrt{6}$.

8. Let

$$\mathbf{n} = (2\mathbf{e}_2 - \mathbf{e}_1) \times (3\mathbf{e}_3 - \mathbf{e}_1) = \begin{pmatrix}6\\3\\2\end{pmatrix}.$$

a. The area of $\mathscr{T}$ is $\frac{1}{2}\|\mathbf{n}\| = \frac{7}{2}$.

b. The plane containing $\mathscr{T}$ has equation $6x + 3y + 2z = 6$ (the left hand side is given by $\mathbf{n}$, and the right hand side by, *e.g.*, the x-intercept).

c. The distance from $3\mathbf{e}_3$ to the line through $\mathbf{e}_1$ and $2\mathbf{e}_2$ is given by

$$\frac{\|\mathbf{n}\|}{\|2\mathbf{e}_2 - \mathbf{e}_1\|} = \frac{7}{5}\sqrt{5}.$$

d. The angle is the acute angle between $\mathbf{n}$ and $\mathbf{e}_3$, or $\arccos\left(\dfrac{|\mathbf{n} \cdot \mathbf{e}_3|}{\|\mathbf{n}\|\|\mathbf{e}_3\|}\right) = \arccos\left(\frac{2}{7}\right)$.

9. a. The centre of $\mathscr{C}$ is $\mathbf{c} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$ (the midpoint of the segment joining $\mathbf{u}$ and $\mathbf{v}$).

b. The radius of $\mathscr{C}$ is $r = \frac{1}{2}\|\mathbf{u} - \mathbf{v}\|$ (half the distance between $\mathbf{u}$ and $\mathbf{v}$).

c. We have

$$\begin{aligned}\|\mathbf{c}\|^2 - r^2 &= \mathbf{c} \cdot \mathbf{c} - r^2\\ &= \tfrac{1}{4}(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) - \tfrac{1}{4}(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})\\ &= \tfrac{1}{4}(\mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}) - \tfrac{1}{4}(\mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v})\\ &= \mathbf{u} \cdot \mathbf{v},\end{aligned}$$

as required.

d. Observe that $\mathbf{p}$ lies on the circle $\mathscr{C}$ if, and only if, $\|\mathbf{p} - \mathbf{c}\| = r$, or equivalently, $(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) = r^2$. Now

$$\begin{aligned}\mathbf{p} \cdot \mathbf{p} &= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{(\mathbf{v} \cdot \mathbf{v})^2}\mathbf{v} \cdot \mathbf{v} = \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v} \cdot \mathbf{v}},\\ 2\mathbf{p} \cdot \mathbf{c} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) = \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v} \cdot \mathbf{v}} + \mathbf{u} \cdot \mathbf{v},\end{aligned}$$

and hence

$$\begin{aligned}(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{p} - \mathbf{c}) &= \mathbf{p} \cdot \mathbf{p} - 2\mathbf{p} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}\\ &= \mathbf{p} \cdot \mathbf{p} - 2\mathbf{p} \cdot \mathbf{c} + \mathbf{u} \cdot \mathbf{v} + r^2 && \text{(by part c)}\\ &= r^2.\end{aligned}$$

Therefore, $\mathbf{p}$ *does* lie on $\mathscr{C}$.

This last part can be done without an explicit calculation, as follows: Let T be the affine isometry that reflects points in the line $\mathbf{c} + \operatorname{Span}\{\mathbf{v}\}$; then T maps $\mathbf{u}$ to $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ and $\mathscr{C}$ to $\mathscr{C}$, from which it follows that $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ lies on $\mathscr{C}$ since $\mathbf{u}$ does.

10. a. $0 \in \mathscr{S}$, since $M0 = 0 = -0M$. Next, if $X, Y \in \mathscr{S}$ and $\alpha, \beta \in \mathbb{R}$ then $M(\alpha X + \beta Y) = \alpha MX + \beta MY = -\alpha XM - \beta YM = -(\alpha X + \beta Y)M$. Therefore $\mathscr{S}$ is closed under linear combinations, so $\mathscr{S}$ is a subspace of the linear space of all 2×2 matrices. $M_{2\times 2}$.

b. Let

$$X = \begin{pmatrix}a & b\\ c & d\end{pmatrix}; \quad \text{then} \quad 0 = MX + XM = \begin{pmatrix}2a + 2c & 2a + 2d\\ 0 & 2c - 2d\end{pmatrix}$$

if, and only if, $a = -c = -d$ (b and d are free variables), and so

$$\left\{\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}$$

is a basis for $\mathscr{S}$. Either of the matrices in this basis will do as a specific non-zero matrix that anti-commutes with M.

c. If A and B anti-commute then

$$A^2B^2 = -ABAB = BAAB = -BABA = B^2A^2.$$

d. If A and B anti-commute then

$$(A + B)^2 = A^2 + AB + BA + B^2 = A^2 + B^2.$$

e. If A and B anti-commute then

$$\begin{aligned}(A + B)^4 &= (A^2 + B^2)^2 && \text{(by part d)}\\ &= A^4 + A^2B^2 + B^2A^2 + B^4\\ &= A^4 + 2A^2B^2 + B^4 && \text{(by part c).}\end{aligned}$$