1. Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 & 2 \\ 2 & 2 & -1 & 3 & 3 \\ 3 & 3 & -1 & 5 & 4 \\ 7 & 7 & -5 & 9 & 12 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

a. Find conditions on b_1, b_2, b_3 and b_4 such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

b. Find the general solution of $A\mathbf{x} = \mathbf{0}$.

c. Find a basis for the column space of A.

d. Find a basis for the null space of A.

e. What is the dimension of the null space of A^T?

2. a. Let $\mathbf{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$.

i. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

ii. Is $\{A\mathbf{v}_1, A\mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

iii. Prove that if A is an invertible 2×2 matrix and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$ then $\{A\mathbf{v}_1, A\mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$.

b. Suppose that N is a square matrix such that $N^3 = 0$.

i. Simplify $(I + N)(I - N + N^2)$.

ii. Find $(I + N)^{-1}$.

iii. Solve the equation $N = (I - X)^{-1}(I + X)$ for X.

3. Given $A = \begin{pmatrix} -8 & 6 \\ -18 & 13 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$.

a. Solve the equation $BY = C$ for Y.

b. Solve the equation $CZ = A$ for Z. If there are infinitely many solutions, give your answer using parameters.

c. Evaluate $B^{-1}AB$.

d. Find four different matrices R such that $R^2 = B^{-1}AB$.

e. Find an elementary matrix E such that EB is lower triangular.

4. a. Given that A is an invertible $m \times m$ matrix, find the inverse of

$$\begin{pmatrix} A & 0 & 0 \\ B & I_n & 0 \\ C & D & I_p \end{pmatrix}$$

(where B is an $n \times m$ matrix, C is a $p \times m$ matrix and D is a $p \times n$ matrix).

b. Balance the chemical equation $Zn + H_2SO_4 \rightarrow H_2S + ZnSO_4 + H_2O$ using matrix methods. (No credit unless you show your work.)

5. a. Solve $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ for x_3 only, using Cramer's rule.

b. Find an LU factorization of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 2 & 0 \end{pmatrix}$.

c. Evaluate $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$.

6. a. Suppose that X and Y are 4×4 matrices, $\det X = -3$ and $\det Y = 2$. Evaluate $\det\big(X^{-1}\operatorname{adj}(Y^T)\big)$.

b. Prove that $\det(A^T - B) = \det(B^T - A)$ if A, B are $n \times n$ matrices and n is even.

c. Find square matrices A and B for which $\det(A + B) \neq \det(A) + \det(B)$.

d. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

Find a basis for $\mathscr{H} = \{\, B \in M_{2\times 2} : \det(A + B) = \det A + \det B \,\}$.

e. Does A belong to $\mathscr{H}$? Explain your answer.

7. Given $\mathbf{p} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

a. Sketch the parallelogram $\mathscr{P}$ formed by $\mathbf{p}$ and $\mathbf{q}$.

b. Give a parametric vector equation for the line ℓ that contains $\mathbf{p}$ and $\mathbf{q}$. Draw ℓ on your sketch from part (a).

c. Find the area of the part of $\mathscr{P}$ that is below ℓ.

d. Find the standard matrix of a rotation R about the origin such that $R(\ell)$ is a horizontal line.

e. Find a vertical shear T such that $T(\mathscr{P})$ is a rectangle. Recall that T is a vertical shear if its standard matrix has the form

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix},$$

where k is a real number.

8. a. Suppose that A is a 5×3 matrix with linearly independent columns. Fill in the missing word in each of the following sentences. The missing word is *must*, *might* or *cannot*.

i. The equation $A^T\mathbf{x} = \mathbf{0}$ ________ have nontrivial solutions.

ii. If $\mathbf{b} \in \mathbb{R}^5$, then the equation $A\mathbf{x} = \mathbf{b}$ ________ be consistent.

iii. The null space of AA^T ________ contain only the zero vector.

iv. $\begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix}$ ________ be invertible.

b. Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and define the linear transformations $P, Q\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$P(\mathbf{x}) = \operatorname{proj}_{\mathbf{v}} \mathbf{x} \quad \text{and} \quad Q(\mathbf{x}) = \operatorname{perp}_{\mathbf{v}} \mathbf{x}.$$

i. Write down the standard matrix of P and the standard matrix of Q.

ii. What is the rank of P?

iii. What is the rank of Q?

iv. $\mathfrak{True}$ or $\mathfrak{false}$: Every vector in the range of Q is in the kernel of P.

9. a. Given $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 7 \\ -2 \\ 7 \end{pmatrix}$.

i. Show that $\mathbf{v}$ is in the span of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

ii. Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ linearly independent? If not, give a specific nontrivial dependence equation satisfied by $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

iii. Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$ linearly independent? If not, give a specific nontrivial dependence equation satisfied by $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}$.

iv. For which values of k is $\{\mathbf{u}_1 + 3\mathbf{u}_3, \mathbf{u}_2 + 5\mathbf{u}_1, \mathbf{u}_3 + k\mathbf{u}_2\}$ linearly independent?

b. Give an example of each of the following if possible. If this is not possible, explain why not.

i. A subset of $\mathbb{R}^2$ that is closed under addition, but not closed under scalar multiplication.

ii. Two three dimensional subspaces of $\mathbb{R}^4$ that intersect in only a line in $\mathbb{R}^4$.

10. Let $\mathscr{P}_1$ be the plane containing

$$\mathbf{p} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix},$$

and let $\mathscr{P}_2$ be the plane with equation $2x + y - z = 6$.

a. Give an equation of the form $ax + by + cz = d$ for $\mathscr{P}_1$.

b. Find the angle between $\mathscr{P}_1$ and $\mathscr{P}_2$.

c. Find the distance from the origin to the $\mathscr{P}_2$.

d. Find the distance from $\mathbf{r}$ to the line that contains $\mathbf{p}$ and $\mathbf{q}$.