1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & -1 & 5 & 11 \\ 5 & 10 & 5 & 10 & -5 & 25 & 55 \\ -2 & -4 & -1 & -1 & -1 & 0 & 2 \\ 1 & 2 & 3 & 8 & -9 & 31 & 75 \\ 3 & 6 & 1 & 0 & 2 & 0 & -5 \end{pmatrix},$$

which row reduces to

$$B = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a. Find the rank of A.

b. Find a basis for, and the dimension of, $\text{col}(A)$.

c. Write the 6^{th} and 7^{th} columns of A as a linear combination of the vectors obtained in part (b).

d. Find a basis for, and the dimension of, $\text{nul}(A)$.

e. Find a basis for $\text{row}(A)$.

f. Write the first column of A^T as a linear combination of the vectors obtained in part (e).

g. What is the dimension of $\text{nul}(A^T)$?

2. Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} 3 \\ 9 \\ 7 \\ 3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ k \\ -3 \\ 2k \end{pmatrix}$, and let $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

a. Give an implicit equation for V.

b. Are the following sets of vectors linearly dependent or independent?
i. $\{\mathbf{u}_1, \mathbf{u}_3\}$ ii. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4\}$ iii. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$

c. For what values of k, if any, is $\mathbf{w}$ in V?

d. Give a basis for V such that none of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ is included in your basis.

3. Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, and define linear transformations $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ by $T_1(\mathbf{x}) = A\mathbf{x}$, and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ by $T_2(\mathbf{x}) = A^T\mathbf{x}$. Also, let $\mathscr{S} = \{\, \mathbf{x} \in \mathbb{R}^2 : 0 \leq x_1 \leqslant 1 \text{ and } 0 \leqslant x_2 \leqslant 1 \,\}$ be the unit square in $\mathbb{R}^2$, and let ℓ be the line in $\mathbb{R}^3$ with parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{u}$.

a. Evaluate is $T_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $T_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

b. Find $T_2 \circ T_1(\mathscr{S})$. Draw pictures of $\mathscr{S}$ and $T_2 \circ T_1(\mathscr{S})$.

c. Find $T_1 \circ T_2(\ell)$.

d. Which of the mappings T_1, T_2, $T_1 \circ T_2$, $T_2 \circ T_1$ is surjective?

e. Which of the mappings T_1, T_2, $T_1 \circ T_2$, $T_2 \circ T_1$ is injective?

4. Let $A = \begin{pmatrix} 1 & 2 & 7 \\ 1 & 0 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 0 \\ -2 & 8 \\ 1 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & 1 \\ -3 & 3 \end{pmatrix}$.

a. Evaluate $AB + 3C$.

b. If possible, find a matrix X such that $3CX = I - ABX$.

c. What is the rank of the 5×5 matrix $\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$?

d. Let Y be an $n \times 2$ matrix. Fill in the blanks with *must*, *might* or *cannot* to make each of the following statements true.

i. If Y has two pivot positions then the mapping $\mathbf{x} \rightsquigarrow YC\mathbf{x}$ ____ be surjective, and the mapping $\mathbf{y} \rightsquigarrow CY^T\mathbf{y}$ ____ be surjective.

ii. If Y has one pivot position then YC ____ have linearly independent columns, and CY^T ____ have linearly independent columns.

5. Let $A = \begin{pmatrix} 1 & 2 & 0 & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -2 & 0 & -3 \\ 1 & -2 & 3 & 4 \end{pmatrix}$.

a. Find the following determinants:
i. $\det(A)$ ii. $\det(-3A)$ iii. $\det(A^{-2})$
iv. $\det(PAP^{-1})$ where P is a 4×4 invertible matrix.
v. $\det(BAB)$ where B is a singular (i.e. non-invertible) matrix.
vi. $\det(D)$ where D is the reduced row echelon form of the matrix A.

b. Use the determinant of A^{-1} to find $\text{adj}(A^{-1})$.

6. Find all values of s for which the following system is inconsistent. For full marks show the work that justifies your answer.

$$\begin{aligned} 3sx_1 + 2x_2 &= 4 \\ 6x_1 + sx_2 &= -4 \end{aligned}$$

7. An $n \times n$ matrix B is called *idempotent* if $B^2 = B$.

a. Suppose that B is an $n \times n$ idempotent matrix
i. Show that $\det B = 0$ or $\det B = 1$.
ii. Show that if $\det B = 1$ then $B = I_n$.
iii. Show that $I_n - B$ is also idempotent.

b. For what values of a and b is $\begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix}$ idempotent?

c. Show that $\begin{pmatrix} A & \frac{1}{k}A \\ k(I_n - A) & I_n - A \end{pmatrix}$ is idempotent for any $n \times n$ matrix A and any non-zero scalar k.

8. Let $V = \left\{ X \in M_{2\times2} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X = X \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

a. Is $0_{2\times2} \in V$? Is $I_2 \in V$?

b. For which values of a, if any, is $\begin{pmatrix} 2 & 2 \\ 3 & a \end{pmatrix} \in V$?

c. Find a basis for V.

d. Write the matrix you found in part (c) as a linear combination of the basis matrices you found in part (d).

9. Determine whether each of the following subsets of $\mathbb{P}_2$ (the space of polynomials of degree at most 2 with real coefficients) are subspaces of $\mathbb{P}_2$. Justify your answers. If a set is a subspace, then give a basis for it.

a. $\{p(x) : p'(1) = 0\}$ b. $\{p(x) : \int_0^1 p(x)\,dx = 1\}$

10. Given $P(0,0,1)$, $Q(1,1,2)$, $R(4,6,5)$ and $S(6,11,10)$, find:

a. a normal to the plane containing P, Q and R;

b. a normal equation of the plane containing P, Q and R;

c. a normal equation of the plane through the origin parallel to the plane found in part (b);

d. the area of triangle PQR;

e. the volume of the parallelepiped three of whose sides are PQ, PR and PS;

f. the distance between the point S and the plane from part (b).

11. The identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ holds for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

a. Fill in the blanks with *must*, *might* or *cannot* to make each of the following statements true.
i. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ ____ lie in the span of $3\mathbf{v}$ and $5\mathbf{w}$.
ii. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ ____ be orthogonal to $2\mathbf{v} \times (-4\mathbf{w})$.
iii. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ ____ be a solution of $\mathbf{v} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{x} = 0$.
iv. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ ____ be parallel to $\mathbf{u}$.

b. Give specific non-zero vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}$.

c. Use the identity to simplify $(\mathbf{u} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{w})$.

d. Use the identity to write $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

12. Consider the planes $4x + y - 3z = 7$ and $2x - 3y + 3z = 4$.

a. Find their line of intersection.

b. Find the cosine of the angle between the planes.

13. Find the point of intersection of the line

$$\mathbf{x} = \begin{pmatrix} 9 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$$

with the plane containing both the y-axis and the z-axis.

1. a. The rank of A is four.

b. The list of pivot columns of A,

$$\left\{ \begin{pmatrix}1\\5\\-2\\1\\3\end{pmatrix}, \begin{pmatrix}1\\5\\-1\\3\\1\end{pmatrix}, \begin{pmatrix}-1\\-5\\-1\\-9\\2\end{pmatrix}, \begin{pmatrix}5\\25\\0\\31\\0\end{pmatrix} \right\},$$

is a basis for $\operatorname{col} A$. The dimension of $\operatorname{col} A$ is four.

c. The sixth column of A is a pivot column, so we have $\mathbf{a}_6 = \mathbf{a}_6$. To express the seventh column of A as a linear combination of the pivot columns, look at the entries of the corresponding column in the reduced echelon form, B, of A: $\mathbf{a}_7 = 2\mathbf{a}_1 - \mathbf{a}_3 - 5\mathbf{a}_5 + \mathbf{a}_6$.

d. The entries in the non-pivot columns of B (the reduced echelon form of A) show how the non-pivot columns of A are linear combinations of the pivot columns of A, and the coefficients in these linear combinations provide the corresponding entries in each of a list of basis vectors for $\operatorname{nul} A$:

$$\left\{ \begin{pmatrix}-2\\1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-3\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\\0\\5\\-1\\1\end{pmatrix} \right\}.$$

The dimension of $\operatorname{nul} A$ is three.

e. Being linearly independent, and linear combinations of the columns of A^T and vice versa, the non-zero columns of B^T form a basis for $\operatorname{row} A = \operatorname{col} A^T$:

$$\left\{ \begin{pmatrix}1\\2\\0\\-1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}0\\0\\1\\3\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\0\\-5\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\1\\1\end{pmatrix} \right\}.$$

f. The entries in the first row of A that occupy pivot columns are the coefficients of the linear combination of the non-zero columns of B^T that is equal to the first column of A^T: $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 + 5\mathbf{v}_4$, where $\mathbf{u}_1$ is the first column of A^T and $\mathbf{v}_i$ are the non-zero columns of B^T.

g. By the rank formula, $\dim \operatorname{nul} A^T = 5 - \dim \operatorname{col} A^T = 5 - 4 = 1$.

2. Row reducing gives

$$\begin{pmatrix}1 & 2 & 1 & 3 & x_1\\ 3 & 0 & 1 & 9 & x_2\\ 1 & 1 & 2 & 7 & x_3\\ 0 & 0 & 1 & 3 & x_4\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 1 & 3 & x_1\\ 0 & -6 & -2 & 0 & -3x_1 + x_2\\ 0 & 0 & \frac{4}{3} & 4 & -\frac{1}{2}x_1 - \frac{1}{6}x_2 + x_3\\ 0 & 0 & 0 & 0 & \frac{3}{8}x_1 + \frac{1}{8}x_2 - \frac{3}{4}x_3 + x_4\end{pmatrix}.$$

a. The rightmost vector in the foregoing matrix is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$ if, and only if, $3x_1 + x_2 - 6x_3 + 8x_4 = 0$, which gives an implicit equation defining V.

b. From the echelon form of the matrix at the beginning of this problem, it follows that:

i. $\{\mathbf{u}_1, \mathbf{u}_3\}$ is linearly independent;

ii. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4\}$ is linearly independent;

iii. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is not linearly independent.

c. Using the implicit equation from part (a), we find that $\mathbf{w}$ belongs to V if, and only if, $3(1) + 1(k) - 6(-3) + 8(2k) = 0$, *i.e.*, $k = -\frac{21}{17}$.

d. Among many possibilities, $\{-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3\}$ is a basis for V that does not contain any of the $\mathbf{u}_i$.

3. a. We have

$$T_1\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}1 & 1\\ -1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}2\\4\\3\end{pmatrix}$$

and

$$T_2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1 & -1 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}.$$

b. Since

$$T_2 \circ T_1(\mathbf{e}_1\ \mathbf{e}_2) = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix},$$

it follows that $T_2 \circ T_1(\mathscr{S}) = \{\, \mathbf{x} \in \mathbb{R}^2 \colon 0 \leqslant x_1 \leqslant 2 \text{ and } 0 \leqslant x_2 \leqslant 3 \,\}$.

c. Since

$$T_1 \circ T_2\begin{pmatrix}3 & 1\\ 1 & 1\\ 0 & -2\end{pmatrix} = \begin{pmatrix}6 & 0\\ 2 & 0\\ 4 & 0\end{pmatrix}, \text{ it follows that } T_1 \circ T_2(\ell) = \left\{ \begin{pmatrix}6\\2\\4\end{pmatrix} \right\}.$$

d. T_2 and $T_2 \circ T_1$ are surjective.

e. T_1 and $T_1 \circ T_2$ are injective.

4. a. i. $AB + 3C = \begin{pmatrix}9 & 9\\ 2 & 4\end{pmatrix} + \begin{pmatrix}15 & 3\\ -9 & 9\end{pmatrix} = \begin{pmatrix}24 & 12\\ -7 & 13\end{pmatrix}$

ii. The given equation is equivalent to $(AB + 3C)X = I_2$, so

$$X = (AB + 3C)^{-1} = \tfrac{1}{396}\begin{pmatrix}13 & -12\\ 7 & 24\end{pmatrix}.$$

iii. The rank of the given (partitioned) matrix is four.

b. i. If Y has two pivot positions then the mapping $\mathbf{x} \rightsquigarrow YC\mathbf{x}$ *might* be surjective, and the mapping $\mathbf{y} \rightsquigarrow CY^T\mathbf{y}$ *must* be surjective.

ii. If Y has one pivot position then YC *cannot* have linearly independent columns, and CY^T *might* have linearly independent columns.

5. a. i.

$$\det A = \begin{vmatrix}-1 & -1 & 3\\ -2 & 0 & -3\\ -4 & 3 & 9\end{vmatrix} = 3\begin{vmatrix}2 & 3\\ 7 & 1\end{vmatrix} = -57.$$

ii. $\det(-3A) = (-3)^4(-57) = -4617$.

iii. $\det(A^{-2}) = (-57)^{-2} = \frac{1}{3249}$.

iv. $\det(PAP^{-1}) = (\det P)(\det A)(\det P)^{-1} = -57$.

v. $\det(BAB) = (\det B)(\det A)(\det B) = 0$, since B is singular 4×4 matrix (and so $\det B = 0$).

vi. Since A is nonsingular, $D = I_4$, and so $\det D = 1$.

b. $\operatorname{adj}(A^{-1}) = \det(A^{-1})(A^{-1})^{-1} = -\frac{1}{57}A$.

6. The determinant of the coefficient matrix of the given linear system is $3(s^2 - 4)$, which is zero if $s = \pm 2$; otherwise the linear system will have one, and only one, solution. If $s = -2$ then the linear system, *i.e.*, $-6x_1 + 2x_2 = 4$ and $6x_1 - 2x_2 = -4$, has infinitely many solutions. If $s = 2$ then the linear system, *i.e.*, $6x_1 + 2x_2 = 4$ and $6x_1 + 2x_2 = -4$, is inconsistent. Therefore, the given linear system is inconsistent if, and only if, $s = 2$.

7. a. i. If $B^2 = B$ then $(\det B)^2 = \det B$, or $(\det B)(\det B - 1) = 0$, so $\det B$ is zero or one.

ii. Since B is idempotent, $0_{n\times n} = B^2 - B = B(B - I_n)$. If, in addition, $\det B = 1$ then B is invertible, and so $B = I_n$.

iii. If B is idempotent, then $(I_n - B)^2 = I_n - 2B + B^2 = I_n - B$, so $I_n - B$ is idempotent.

b. Since

$$\begin{pmatrix}2 & 3\\ a & b\end{pmatrix}\begin{pmatrix}2 & 3\\ a & b\end{pmatrix} = \begin{pmatrix}4 + 3a & 6 + 3b\\ ab + 2a & 3a + b^2\end{pmatrix},$$

if the given matrix is idempotent then $4 + 3a = 2$, so $a = -\frac{2}{3}$, and $6 + 3b = 3$, so $b = -1$. To be sure that the given matrix is idempotent, it is necessary to check that then $ab + 2a(-\frac{2}{3})(-1) + 2(-\frac{2}{3}) == -\frac{2}{3} = a$ (ok) and $3a + b^2 = 3(-\frac{2}{3}) + (-1)^2 = -1 = b$ (ok).

c. If A is any $n \times n$ matrix and k is any non-zero scalar, then

$$\begin{aligned}
&\begin{pmatrix} A & \frac{1}{k}A \\ k(I_n - A) & I_n - A \end{pmatrix} \begin{pmatrix} A & \frac{1}{k}A \\ k(I_n - A) & I_n - A \end{pmatrix} \\
&= \begin{pmatrix} A^2 + A(I_n - A) & \frac{1}{k}A^2 + \frac{1}{k}A(I_n - A) \\ k(I_n - A)A + k(I_n - A)^2 & (I_n - A)A + (I_n - A)^2 \end{pmatrix} \\
&= \begin{pmatrix} A & \frac{1}{k}A \\ k(I_n - A) & I_n - A \end{pmatrix}
\end{aligned}$$

since $A^2 + A(I_n - A) = A^2 + A - A^2 = A$, and $(I_n - A)A + (I_n - A)^2 = A - A^2 + I_n - 2A + A^2 = I_n - A$.

8. If

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \quad \text{and} \quad X \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix},$$

which are equal if, and only if, $b = a$ and $d = -c$; *i.e.*,

$$X = \begin{pmatrix} a & a \\ c & -c \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

for some real numbers a and c.

a. $0_{2\times 2}$ belongs to V and I_2 does not belong to V.

b. The matrix

$$\begin{pmatrix} 2 & 2 \\ 3 & a \end{pmatrix}$$

belongs to V if, and only if, $a = -3$.

c. From the explicit description of V given above, it follows that

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

is a basis for V (since the matrices in this list are clearly linearly independent).

d. The matrix in part (c) is equal to

$$\begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

9. a. This set, call it $\mathscr{V}$, is a subspace of $\mathbb{P}_2$ since it is the kernel of a linear transformation (namely, the composite of differentiation and evaluation at 1). If $p(t) = a + bt + ct^2$ then $p'(1) = b + 2c$, which is equal to zero if, and only if, $b = -2c$, and so $p(t) = a + c(-2t + t^2)$. Thus, $\{1, -2t + t^2\}$ (since it is clearly linearly independent) is a basis for $\mathscr{V}$.

b. Since $\int_0^1 0\,dt = 0 \neq 1$, this set does not contain the zero polynomial, and so it is not a subspace of $\mathbb{P}_2$.

10. We have

$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \overrightarrow{PR} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} \quad \text{and} \quad \overrightarrow{PS} = \begin{pmatrix} 6 \\ 11 \\ 9 \end{pmatrix}.$$

a. A normal to the plane containing P, Q and R is

$$\mathbf{n} = -\tfrac{1}{2}\overrightarrow{PQ} \times \overrightarrow{PR} = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

b. A normal equation for the plane containing P, Q and R is $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\overrightarrow{OP}$, or $x - z = -1$ (where O is the origin in $\mathbb{R}^3$).

c. A normal equation for the plane that is parallel to the plane from part (b) and contains the origin is $x - z = 0$ (only the right hand side changes).

d. The area of the triangle PQR is half the area of the parallelogram formed by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, *i.e.*, $\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{2}$.

e. The volume of the parallelepiped three of whose sides are $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ is equal to $\left|\det\left(\overrightarrow{PQ}\ \overrightarrow{PR}\ \overrightarrow{PS}\right)\right| = \left|\left(\overrightarrow{PQ} \times \overrightarrow{PR}\right) \cdot \overrightarrow{PS}\right| = 6$.

f. The distance from S to the plane from part (b) is equal to the volume from part (e) divided by the twice the area from part (d), *i.e.*, $\frac{3}{2}\sqrt{2}$.

11. a. i. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *must* lie in the span of $3\mathbf{v}$ and $5\mathbf{w}$.

ii. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *must* be orthogonal to $2\mathbf{v} \times (-4\mathbf{w})$.

iii. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *might* be a solution of $\mathbf{v} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{x} = 0$.

iv. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *cannot* be parallel to $\mathbf{u}$ (under the convention that no vector is parallel to the zero vector).

b. Using the given identity, or otherwise, one finds that $\mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{e}_1) = \mathbf{e}_2$.

c.

$$\begin{aligned}
(\mathbf{u} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{w}) &= \big((\mathbf{u} \times \mathbf{w}) \cdot \mathbf{w}\big)\mathbf{v} - \big((\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}\big)\mathbf{w} \\
&= -\big((\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}\big)\mathbf{w} \\
&= -\det\big(\mathbf{u}\ \ \mathbf{w}\ \ \mathbf{v}\big)\mathbf{w} \\
&= \det\big(\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}\big)\mathbf{w}
\end{aligned}$$

d.

$$\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &= -\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) \\
&= -\big((\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v}\big) \\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}
\end{aligned}$$

12. a. The line of intersection of the planes is found by solving the system of their (linear) equations. Now,

$$\begin{pmatrix} 2 & -3 & 3 & 4 \\ 4 & 1 & -3 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{3}{7} & \frac{25}{14} \\ 0 & 1 & -\frac{9}{7} & -\frac{1}{7} \end{pmatrix},$$

and so the line of intersection has parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{25}{14} \\ -\frac{1}{7} \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 9 \\ 7 \end{pmatrix}.$$

b. The cosine of the (acute) angle between the planes is equal to

$$\frac{|(4)(2) + (1)(-3) + (-3)(3)|}{\sqrt{4^2 + 1^2 + (-3)^2}\sqrt{2^2 + (-3)^2 + 3^2}} = \frac{4}{\sqrt{26}\sqrt{22}} = \tfrac{2}{143}\sqrt{143}.$$

13. The given line meets the yz-plane where $x = 9 + 5t = 0$, or $t = -\frac{9}{5}$, so its point of intersection with the yz-plane has coordinate vector

$$\begin{pmatrix} 9 \\ -1 \\ 3 \end{pmatrix} - \tfrac{9}{5} \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{14}{5} \\ \frac{6}{5} \end{pmatrix}.$$