1. Given the matrix $A = \begin{pmatrix} 1 & 4 & -1 & 1 \\ -2 & -8 & -2 & 6 \\ 3 & 12 & -1 & -1 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$.

a. Solve the system $A\mathbf{x} = \mathbf{b}$. b. Solve the system $A\mathbf{x} = \mathbf{0}$.

2. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined as follows.

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2 + x_3 \\ 2x_3 - x_4 \end{pmatrix}$$

a. Find the standard matrix of T.
b. Is T injective? Justify your answer.
c. Is T surjective? Justify your answer.

3. Find all values of λ for which the system of linear equations

$$\begin{aligned} x + y - z &= 0 \\ x + (\lambda+1)y + 2z &= 0 \\ x + y + (\lambda-5)z &= 0 \end{aligned}$$

has a. no solutions, b. a unique solution, c. infinitely many solutions.

4. Find a second degree polynomial whose graph contains the points $(1,1)$ and $(2,6)$ and whose derivative at 1 is 4.

5. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$ find A^{-1}.

6. Find and LU factorization of the matrix $\begin{pmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{pmatrix}$.

7. Find the inverse of the partitioned matrix $\begin{pmatrix} I & M \\ N & 0 \end{pmatrix}$ given that M and N are invertible $n \times n$ matrices.

8. Given $A = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$, find $(AA^T)^{-1}$.

9. Given that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $\det(A) = 5$, find:

a. $\det(4A)$, b. $\det(AA^T)$, c. $\det(\operatorname{adj} A)$, d. $\begin{vmatrix} a & b & c \\ g+3a & h+3b & i+3c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \end{vmatrix}$.

10. Suppose that A, B and C are $n \times n$ matrices and $ABC = I$. Find B^{-1}.

11. If A is a 9×9 matrix such that $A^T = -A$ then prove that $\det A = 0$. Is the same result true for a 10×10 matrix A? Justify your answer completely (with a proof, or counterexample, as appropriate).

12. Given $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 5 & 2 & -7 & 3 \\ 3 & 0 & 6 & 2 \\ 5 & 2 & -4 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

a. Solve $A\mathbf{x} = \mathbf{b}$ for x_3 only using Cramer's rule.
b. How many solutions does $A\mathbf{x} = \mathbf{0}$ have?

13. Are the following true or false? (All matrices are $n \times n$.) Justify your answer. No credit will be given without justification.

a. $\det(E_1E_2) \neq 0$ if E_1 and E_2 are elementary matrices.
b. $(A-B)(A+B) = A^2 - B^2$.
c. $\det(I + A) = 1 + \det A$.
d. If $S\colon \mathbb{R}^n \to R^n$ are linear transformations such that S is surjective and T is not surjective, then $S \circ T$ is not surjective.
e. The non-pivot columns of a matrix form a linearly dependent list.

14. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which scales every vector by 5 then reflects the vector through the y-axis and finally rotates the vector about the origin by $\frac{1}{4}\pi$ radians clockwise. Find the standard matrix of T.

15. Let A be a 7×9 matrix of rank 4.

a. What is $\dim \operatorname{Nul} A$? b. What is $\dim \operatorname{Row} A$?
c. What is $\operatorname{rank}(A^T)$? d. What is $\dim \operatorname{Nul}(A^T)$?

16. Given $A = \begin{pmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

a. Find a basis for, and the dimension of, the column space of A.
b. Write every column of A not in this basis as a linear combination of the basis vectors.
c. Find a basis for, and the dimension of, the null space of A.
d. Find a basis for the row space of A.
e. Do the columns of A span $\mathbb{R}^4$?

17. Find a basis for, and the dimension of, each of the following vector spaces.

a. $\mathscr{S} = \{\, A \in M_{2\times 2} \colon A^T = A \,\}$

b. $\mathscr{K} = \{\, \mathbf{x} \in \mathbb{R}^3 \colon \mathbf{x}$ is orthogonal to $\mathbf{a} \,\}$, where $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

c. $\mathscr{L} = \{\, p(t) \in \mathbb{P}_3 \colon p(0) = 0 \,\}$

18. Let $\mathscr{Q} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \colon x \leqslant 0 \text{ and } y \leqslant 0 \right\}$.

a. Does $\mathscr{Q}$ contain the zero vector? b. Is $\mathscr{Q}$ closed under scaling?
c. Is $\mathscr{Q}$ closed under addition? d. Is $\mathscr{Q}$ a subspace of $\mathbb{R}^2$?

19. Determine whether each set is a subspace of the indicated vector space. Justify your answers.

a. $\mathscr{S} = \{\, A \in M_{3\times 3} \colon \det A = 0 \,\}$ in $M_{3\times 3}$.

b. $\mathscr{H} = \left\{ X \in M_{2\times 3} \colon \begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix} X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ in $M_{2\times 3}$.

20. Let $\mathbf{p} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and ϖ the plane defined by $x - 2y + 2z = 0$.

a. Find a basis for the intersection of ϖ and the xy-plane.
b. Find the intersection of ϖ and the line $\mathbf{p} + t\mathbf{u}$.
c. Find the distance from $\mathbf{p}$ to ϖ.
d. Give a parametric vector equation for the line which contains the origin and intersects ϖ at a right angle.

21. Consider the three points $A(1,2,3)$, $B(2,3,1)$ and $C(3,1,2)$.

a. Find the area of the triangle with vertices A, B and C.
b. Find an equation of the plane containing A, B and C.
c. Find the distance from A to the line containing B and C.
d. Does there exist a rotation that carries A to B, B to C and C to A? If so, find the standard matrix of this rotation, and give its axis, angle and orientation. If not, explain why there can be no such rotation.

22. Simplify $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) + \mathbf{i} \times (\mathbf{j} \times \mathbf{k}) + \mathbf{j} \times (\mathbf{i} \times \mathbf{i})$ as much as possible.

23. Write $A = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}$ as a product of elementary matrices.

24. Let $T\colon V \to W$ be a linear transformation, and let $\{v_1, \ldots, v_p\}$ be a linearly dependent list in V. Prove that $\{T(v_1), \ldots, T(v_p)\}$ is linearly dependent.

25. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be a linearly independent list of vectors in a vector space V. Prove that if n is odd then $\{v_1 + v_2, v_2 + v_3, \ldots, v_{n-1} + v_n, v_n + v_1\}$ is linearly independent. Does the same conclusion hold if n is even? If so, give a proof. If not, give a counterexample.

1. a. Row reducing the augmented matrix of $A\mathbf{x} = \mathbf{b}$ gives

$$\begin{pmatrix} 1 & 4 & -1 & 1 & 2 \\ -2 & -8 & -2 & 6 & -4 \\ 3 & 12 & -1 & -1 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 0 & -4 & 8 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so the general solution of $A\mathbf{x} = \mathbf{b}$ consists of all vectors of the form $\mathbf{p} + s\mathbf{u} + t\mathbf{v}$, where s and t are real numbers, and

$$\mathbf{p} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

b. The solution set of $A\mathbf{x} = \mathbf{0}$ is the set of all vectors of the form $s\mathbf{u} + t\mathbf{v}$, where s and t are real numbers, and $\mathbf{u}$ and $\mathbf{v}$ are from part a.

2. a. The standard matrix of T is

$$\big[T\big] = \big(T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3) \quad T(\mathbf{e}_4)\big) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

b. T is not injective because not every column of $\big[T\big]$ is a pivot column (so, *e.g.*, $T(-\mathbf{e}_2 + \mathbf{e}_3 + 2\mathbf{e}_4) = \mathbf{0}$).

c. T is surjective, because $\big[T\big]$ has a pivot position in every row.

3. If A is the coefficient matrix of this *homogeneous* linear system, then

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & \lambda+1 & 2 \\ 1 & 1 & \lambda-5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & \lambda & 3 \\ 0 & 0 & \lambda-4 \end{pmatrix}.$$

a. The given linear system is never inconsistent, since $\mathbf{0}$ is always a solution.

b. The given linear system has a unique solution if $\lambda \neq 0$ and $\lambda \neq 4$ (so that each column of A is a pivot column).

c. The given linear system has infinitely many solutions if $\lambda = 0$ or $\lambda = 4$ (so that at least one of the second and third columns of A is not a pivot column).

4. If $p(t) = x_0 + x_1 t + x_2 t^2$, then $p'(t) = x_1 + 2x_2 t$, and the given conditions yield a system of linear equations in x_0, x_1 and x_2 as follows.

$$\begin{aligned} p(1) = 1 &: \quad x_0 + x_1 + x_2 = 1 \\ p(2) = 6 &: \quad x_0 + 2x_2 + 4x_2 = 6 \\ p'(1) = 4 &: \quad x_1 + 2x_2 = 4 \end{aligned}$$

Row reducing the augmented matrix of this linear system gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \\ 0 & 1 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and so $x_0 = -2$, $x_1 = 2$ and $x_2 = 1$. Therefore, $p(t) = -2 + 2t + t^2$.

5. Row reducing $\begin{pmatrix} A & I_3 \end{pmatrix}$ gives

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 3 & 0 & -3 & 3 & -3 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{pmatrix};$$

therefore,

$$A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}.$$

6. One has

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

via the rough work

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 \end{pmatrix}.$$

7. It is required to find $n \times n$ matrices A, B, C and D such that

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & M \\ N & 0 \end{pmatrix} = \begin{pmatrix} A + BN & AM \\ C + DN & CM \end{pmatrix}.$$

Equating the top right blocks gives $AM = 0$, or $A = 0$, and equating the bottom right blocks gives $CM = I$, or $C = M^{-1}$ (both since M is nonsingular). Next, equating the top left blocks gives $BN = 0$, or $B = N^{-1}$ (since $A = 0$ and N is nonsingular), and equating the bottom left blocks gives $M^{-1} + DN = 0$, or $D = -M^{-1}N^{-1}$ (since $C = M^{-1}$ and N is nonsingular). Therefore (by a little checking, and the first corollary to the Invertible Matrix Theorem),

$$\begin{pmatrix} I & M \\ N & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & N^{-1} \\ M^{-1} & -M^{-1}N^{-1} \end{pmatrix}.$$

8. Notice that $(AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^T A^{-1}$, (and that $(AA^T)^{-1}$ is symmetric, so it suffices to compute two-thirds of its entries), and so

$$(AA^T)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 10 & 10 \\ 10 & 34 & 33 \\ 10 & 33 & 34 \end{pmatrix}.$$

9. a. $\det(4A) = 4^3 \det A = 64 \cdot 5 = 320$, since A is a 3×3 matrix.

b. $\det(AA^T) = (\det A)(\det A^T) = (\det A)^2 = 5^2 = 25$, as the determinant preserves products and is invariant under transposition.

c. Since $(\det A)I_3 = A(\operatorname{adj} A)$ and the determinant preserves products, one has $(\det A)^3 = \big(\det A\big)\big(\det(\operatorname{adj} A)\big)$, and so $\det(\operatorname{adj} A) = (\det A)^2 = 5^2 = 25$.

d. Since the determinant preserves products, one has (*e.g.*, by expanding the left factor along the first or third row, or the second or third column)

$$\begin{vmatrix} a & b & c \\ g + 3a & h + 3b & i + 3c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \big(-\tfrac{1}{2}\big)\big(5\big) = -\tfrac{5}{2}.$$

10. If $ABC = I$ then A is nonsingular (by the first corollary to the Invertible Matrix Theorem), so $I = A^{-1}A = A^{-1}ABCA = BCA$, from which it follows (by the same corollary) that $B^{-1} = CA$.

11. Since $\det A = \det A^T = \det(-A) = (-1)^9 \det A = -\det A$, it follows that $\det A = 0$. While *this particular argument* does not go through if A is a 10×10 matrix, one requires a counterexample to show that there is not a more clever argument for a 10×10 matrix. A simple counterexample is furnished by the block diagonal matrix A whose five 2×2 diagonal blocks are equal to

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $A^T = -A$ since $M^T = -M$, and $\det A = (\det M)^5 = 1$, as required.

12. a. Since

$$\det A = \begin{vmatrix} 1 & 0 & 2 & -1 \\ 5 & 2 & -7 & 3 \\ 3 & 0 & 6 & 2 \\ 5 & 2 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & -1 \\ 5 & 2 & -7 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 3 & -1 \end{vmatrix} = -10 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -30,$$

and

$$\det A_3(\mathbf{b}) = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 5 & 2 & 1 & 3 \\ 3 & 0 & 0 & 2 \\ 5 & 2 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 \\ 3 & 0 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 10,$$

it follows that $x_3 = -\frac{1}{3}$.

b. Since A is nonsingular (because $\det A \neq 0$), it follows that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

13. a. Since $\det(E_1 E_2) = (\det E_1)(\det E_2)$ and E_1, E_2 are invertible (recall that elementary matrices are the building blocks of invertible matrices), it is *true* that $\det(E_1 E_2) \neq 0$.

b. If

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then $A^2 = B^2 = 0$, but $A + B$ and $A - B$ are invertible, so the statement is *false*. (Alternatively, by expanding the putative identity one obtains $AB = BA$, which is falsified by the given matrices.)

c. Since $\det(I_2 + I_2) = \det(2I_2) = 4 \neq 2 = 1 + \det I_2$, this statement is *false*.

d. Since $\operatorname{rank}(S \circ T) \leqslant \operatorname{rank}(T) < n$, $S \circ T$ is not surjective, and therefore this statement is *true*.

e. The statement is trivially *false*. For example, the list of non-pivot columns of an invertible matrix (for example, I_2) is the empty list, which is linearly independent. (Less trivially, the list of non-pivot columns of the 2×2 matrix $\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_1 \end{pmatrix}$ is $\{\mathbf{e}_1\}$, which is linearly independent.)

14. The transformation in question is given by $T = T_3 \circ T_2 \circ T_1$, where T_1 scales vectors by a factor of 5, T_2 reflects vectors in the y-axis (*i.e.*, the $\mathbf{e}_2$-axis), and T_3 rotates vectors about the origin through $-\frac{1}{4}\pi$ radians ($-\frac{1}{4}\pi$, since the rotation is in the clockwise direction). Now,

$$T_1(\mathbf{x}) = 5\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^2, \quad \text{and so} \quad \left[T_1\right] = 5I_2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

Next,

$$T_2(\mathbf{e}_1) = -\mathbf{e}_1 \quad \text{and} \quad T_2(\mathbf{e}_2) = \mathbf{e}_2, \quad \text{and so} \quad \left[T_2\right] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally

$$T_3(\mathbf{e}_1) = \cos(-\tfrac{1}{4}\pi)\mathbf{e}_1 + \sin(-\tfrac{1}{4}\pi)\mathbf{e}_2 = \tfrac{1}{2}\sqrt{2}\mathbf{e}_1 - \tfrac{1}{2}\sqrt{2}\mathbf{e}_2$$

and

$$T_3(\mathbf{e}_2) = -\sin(-\tfrac{1}{4}\pi)\mathbf{e}_1 + \cos(-\tfrac{1}{4}\pi)\mathbf{e}_2 = \tfrac{1}{2}\sqrt{2}\mathbf{e}_1 + \tfrac{1}{2}\sqrt{2}\mathbf{e}_2,$$

and therefore

$$\left[T_3\right] = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} = \tfrac{1}{2}\sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Therefore,

$$\left[T\right] = \left[T_3\right]\left[T_2\right]\left[T_1\right] = \tfrac{1}{2}\sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \tfrac{5}{2}\sqrt{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

15. a. By the rank formula, $\dim \operatorname{Nul} A = 9 - \operatorname{rank} A = 9 - 4 = 5$.

b. Since $\operatorname{Row} A = \operatorname{Col}(A^T)$, $\dim \operatorname{Row} A = \operatorname{rank}(A^T) = \operatorname{rank} A = 4$.

c. $\operatorname{rank}(A^T) = \operatorname{rank} A = 4$.

d. By the rank formula $\dim \operatorname{Nul}(A^T) = 7 - \operatorname{rank}(A^T) = 7 - 4 = 3$.

16. a. A basis for $\operatorname{Col} A$ is given by the list of pivot columns of A, *i.e.*, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$, where $\mathbf{a}_i$ denotes row i of A, for $1 \leqslant i \leqslant 5$; the dimension of $\operatorname{Col} A$ is 3.

b. Since the reduced echelon form of A has the same null space as A, we have $\mathbf{a}_3 = \mathbf{a}_1 + 2\mathbf{a}_2$, and $\mathbf{a}_5 = -\mathbf{a}_1 + 3\mathbf{a}_2 + 4\mathbf{a}_4$.

c. Rewriting the equations from the previous part as dependence equations on the columns of A gives the basis

$$\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$$

of $\operatorname{Nul} A$; the dimension of $\operatorname{Nul} A$ is 2.

d. A basis for $\operatorname{Row} A = \operatorname{Col}(A^T)$ is formed by the transposes of the non-zero rows of the reduced echelon form of A, *i.e.*,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{pmatrix} \right\}$$

is a basis for $\operatorname{Row} A$.

e. Since $\dim \operatorname{Col} A = 3 < 4$, the columns of A *do not* span $\mathbb{R}^4$.

17. a. A matrix $A \in M_{2\times 2}$ belongs to $\mathscr{S}$ if, and only if, there are real numbers a, b and c, such that

$$\begin{aligned} A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} &= a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= aS_1 + bS_2 + cS_3. \end{aligned}$$

Therefore, $\{S_1, S_2, S_3\}$ (which is evidently linearly independent) is a basis for $\mathscr{S}$ (the 2×2 *symmetric* matrices), and the dimension of $\mathscr{S}$ is 3.

b. A vector $\mathbf{x} \in \mathbb{R}^3$ belongs to $\mathscr{K}$ if, and only if, its entries x_1, x_2, x_3 satisfy the homogeneous linear equation $3x_1 + x_2 - 2x_3 = 0$, in which $x_2 = 3s$ and $x_3 = 3t$ are free, and $x_1 = -s + 2t$; so $\mathbf{x}$ has the form $s\mathbf{u} + t\mathbf{v}$, where

$$\mathbf{u} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

and s and t are real numbers. Therefore, $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathscr{K}$, and the dimension of $\mathscr{K}$ is 2.

c. A polynomial $p(t) = x_0 + x_1 t + x_2 t^2 + x_3 t^3 \in \mathbb{P}_3$ belongs to $\mathscr{L}$ if, and only if, $x_0 = 0$; therefore, $\{t, t^2, t^3\}$ is a basis for $\mathscr{L}$, and the dimension of $\mathscr{L}$ is 3.

18. a. Since $0 \leqslant 0$, the zero vector does belong to $\mathscr{Q}$.

b. Since $-\mathbf{e}_1 \in \mathscr{Q}$, but $(-1)(-\mathbf{e}_1) = \mathbf{e}_1 \notin \mathscr{Q}$, $\mathscr{Q}$ is not closed under scaling.

c. Since the sum of two non-positive real numbers is non-positive, it follows that $\mathscr{Q}$ is closed under addition.

d. Since $\mathscr{Q}$ is not closed under scaling, $\mathscr{Q}$ is not a subspace of $\mathbb{R}^2$.

19. a. The 3×3 matrices $A = \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} & \mathbf{0} \end{pmatrix}$ and $B = \begin{pmatrix} \mathbf{0} & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix}$ belong to $\mathscr{S}$, but $A + B = I_3$ does not belong to $\mathscr{S}$, so $\mathscr{S}$ is not closed under addition. Therefore, $\mathscr{S}$ is not a subspace of $M_{3\times 3}$.

b. If $T\colon M_{2\times 3} \to M_{2\times 3}$ be the linear transformation defined by $T(X) = AX$, where

$$A = \begin{pmatrix} 1 & 2 \\ 8 & 16 \end{pmatrix},$$

then $\mathscr{H}$ is the kernel of T, and so $\mathscr{H}$ is a subspace of $M_{2\times 3}$.

20. a. The intersection of ϖ and the xy-plane is the solution set of $x-2y+2z=0$ and $z=0$, in which $y=t$ is free, $x=2t$ and $z=0$; thus, a basis for this line through the origin is

$$\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}\right\}.$$

b. Where ϖ meets the line $\mathbf{p}+t\mathbf{u}$ one has $(3+t)-2(-1+t)+2(t)=0$, or $t=-5$, so the point of intersection is given by

$$\begin{pmatrix}3\\-1\\0\end{pmatrix}-5\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\-6\\-5\end{pmatrix}.$$

c. The distance from $\mathbf{p}$ to ϖ is the length of the projection of $\mathbf{p}$ onto the given normal $\mathbf{n}$ to ϖ (whose entries are the coefficients in the equation defining ϖ), *i.e.*,

$$\frac{|\mathbf{n}^T\mathbf{p}|}{\|\mathbf{n}\|}=\frac{|3-2(-1)+2(0)|}{\sqrt{1^2+(-2)^2+2^2}}=\tfrac{5}{3}.$$

d. The line through the origin which is orthogonal to ϖ consists of all vectors of the form $\mathbf{x}=t\mathbf{n}$, where t is a real number and

$$\mathbf{n}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$$

is the given normal to ϖ.

21. a. The area of the triangle ABC is half the area of the parallelogram formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, *i.e.*,

$$\tfrac{1}{2}\|\overrightarrow{BA}\times\overrightarrow{BC}\|=\tfrac{1}{2}\left\|\begin{pmatrix}-1\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\|=\tfrac{1}{2}\left\|\begin{pmatrix}3\\3\\3\end{pmatrix}\right\|=\tfrac{3}{2}\sqrt{3}.$$

b. A normal to the plane containing A, B and C is given by

$$\mathbf{n}=\tfrac{1}{3}\overrightarrow{BA}\times\overrightarrow{BC}=\begin{pmatrix}1\\1\\1\end{pmatrix},$$

and so a cartesian equation of this plane is given by $\mathbf{n}^T\mathbf{x}=\mathbf{n}^T\overrightarrow{OA}$, or $x+y+z=6$.

c. The distance from A to the line containing B and C is equal to the area of the parallelogram formed by $\overrightarrow{BA}$ and $\overrightarrow{BC}$ divided by the length of $\overrightarrow{BC}$, *i.e.*,

$$\frac{\|\overrightarrow{BA}\times\overrightarrow{BC}\|}{\|\overrightarrow{BC}\|}=\frac{3\sqrt{3}}{\sqrt{6}}=\tfrac{3}{2}\sqrt{2}.$$

d. Observe that

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}=\begin{pmatrix}2&3&1\\3&1&2\\1&2&3\end{pmatrix},$$

and that

$$\begin{vmatrix}1&2&3\\2&3&1\\3&1&2\end{vmatrix}=-18;$$

therefore,

$$P=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

is the standard matrix of the unique linear transformation $T\colon\mathbb{R}^3\to\mathbb{R}^3$ which carries A to B, B to C and C to A. Since

$$P^TP=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I_3$$

and

$$\det P=\begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix}=1,$$

it follows that P is the standard matrix of a rotation in $\mathbb{R}^3$. Next, $P^2=P^T\neq I_3$ and $P^3=P^TP=I_3$ so it follows that the angle of the rotation defined by P is $\frac{2}{3}\pi$. Since P defines a rotation about the line $t\mathbf{n}$, where $\mathbf{n}=\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$, to determine the orientation of the rotation it suffices to find the orientation of the basis $\{\mathbf{x},P\mathbf{x},\mathbf{n}\}$ for $\mathbb{R}^3$, where $\mathbf{x}$ is any non-zero vector in the subspace (plane through the origin) ϖ that is orthogonal to $\mathbf{n}$. Now, $\mathbf{x}=\mathbf{e}_1-\mathbf{e}_2$ is in ϖ, $P\mathbf{x}=-\mathbf{e}_1+\mathbf{e}_3$, and

$$\det\begin{pmatrix}\mathbf{x}&P\mathbf{x}&\mathbf{n}\end{pmatrix}=\begin{vmatrix}1&-1&1\\-1&0&1\\0&1&1\end{vmatrix}=-3.$$

Therefore, the rotation defined by P is oriented so that $\{\mathbf{x},P\mathbf{x},\mathbf{n}\}$ is a left-handed basis for $\mathbb{R}^3$.

22. Using $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$, $\mathbf{k}\times\mathbf{i}=\mathbf{j}$, and that $\mathbf{x}\times\mathbf{x}=\mathbf{0}$ for any $\mathbf{x}\in\mathbb{R}^3$ (from which it follows that $\mathbf{x}\times\mathbf{0}=\mathbf{0}$ and $\mathbf{x}\times\mathbf{y}=-\mathbf{y}\times\mathbf{x}$ for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$), one obtains

$$\begin{aligned}\mathbf{i}\times(\mathbf{i}\times\mathbf{j})+\mathbf{i}\times(\mathbf{j}\times\mathbf{k})+\mathbf{j}\times(\mathbf{i}\times\mathbf{i})&=\mathbf{i}\times\mathbf{k}+\mathbf{i}\times\mathbf{i}+\mathbf{0}\\&=-\mathbf{j}+\mathbf{0}\\&=-\mathbf{j}.\end{aligned}$$

23. A can be obtained from I_2 by first scaling row 2 by a factor of 3, then adding 4 times row 1 to row 2, and finally interchanging rows 1 and 2. Therefore,

$$A=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\4&1\end{pmatrix}\begin{pmatrix}1&0\\0&3\end{pmatrix}.$$

24. If $\{v_1,\dots,v_p\}$ is linearly dependent, then there are scalars $\alpha_1,\dots,\alpha_p$ *not all zero*, such that

$$0_V=\alpha_1v_1+\cdots+\alpha_pv_p,$$

where 0_V is the zero vector of V. Applying T to both sides of this equation and using the linearity of T gives

$$\begin{aligned}\mathbf{0}_W=T(\mathbf{0}_V)&=T(\alpha_1v_1+\cdots+\alpha_pv_p)\\&=\alpha_1T(v_1)+\cdots+\alpha_pT(v_p).\end{aligned}$$

Since the scalars $\alpha_1,\dots,\alpha_p$ are not all zero, it follows that $\{T(v_1),\dots,T(v_p)\}$ is linearly dependent.

25. Let $\alpha_1,\dots\alpha_n$ scalars such that

$$\alpha_1(v_1+v_2)+\alpha_2(v_2+v_3)+\cdots+\alpha_n(v_n+v_1)=0_V \qquad (\dagger)$$

(where 0_V is the zero vector of V), or

$$(\alpha_n+\alpha_1)v_1+(\alpha_1+\alpha_2)v_2+\cdots+(\alpha_{n-1}+\alpha_n)v_n=0_V.$$

Since $\{v_1,\dots,v_n\}$ is linearly independent, it follows that

$$\alpha_n+\alpha_1=0,\quad \alpha_1+\alpha_2=0,\quad \dots,\quad \alpha_{n-1}+\alpha_n=0,$$

and thus $\alpha_2=-\alpha_1$, $\alpha_3=-\alpha_2=\alpha_1$, *&c.*, so that in general $\alpha_i=(-1)^{i-1}\alpha_1$ for $i=2,\dots,n$. Since n is odd, $\alpha_n=\alpha_1$, and since $\alpha_n=-\alpha_1$ (from the first equation displayed above), it follows that $\alpha_i=0$ for $i=1,\dots,n$. Therefore, $\{v_1+v_2,v_2+v_3,\dots,v_n+v_1\}$ is linearly independent.

If n is even then the preceding argument yields $\alpha_1=\alpha_1$ which does not give additional information about α_1. However, it does suggest that for any $\alpha_1\neq 0$, defining $\alpha_i=(-1)^{i-1}\alpha_1$ for $i=2,\dots,n$ will give a non-trivial solution to (†). Indeed this is the case, for if $n=2k$ then the left hand side of (†) becomes

$$\begin{aligned}\alpha_1(v_1+v_2)-\alpha_1(v_2+v_3)+\alpha_1(v_3+v_4)+\cdots-\alpha_1(v_{2k}+v_1)\\=\alpha_1\{(v_1+\cdots+v_{2k})-(v_1+\cdots+v_{2k})\}\\=0_V,\end{aligned}$$

regardless of the values of $v_1,\dots,v_{2k}$. So, *e.g.*, if $v_i=\mathbf{e}_i$ for $i=1,\dots,2k$ in $\mathbb{R}^{2k}$, then $\{v_1+v_2,v_2+v_3,\dots,v_n+v_1\}$ is linearly dependent.