1. Solve the linear system

$$\begin{aligned} 2x_1 - 4x_2 - 2x_3 + 8x_4 &= -4 \\ -3x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\ -x_1 + 3x_2 + 3x_3 - 9x_4 &= 5. \end{aligned}$$

2. Given the matrix $A = \begin{pmatrix} 3 & 5 & 0 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & k & 0 & 1 \\ 0 & 4 & 0 & k \end{pmatrix}$,

a. evaluate $\det A$, and

b. find all values of k such that A is singular.

3. Find the inverse of $A = \begin{pmatrix} 2 & -3 & -14 \\ 1 & -2 & -7 \\ -3 & 5 & 22 \end{pmatrix}$.

4. Find an LU factorization of $A = \begin{pmatrix} 3 & 2 & 6 \\ 9 & 4 & 22 \\ -12 & -12 & -11 \end{pmatrix}$. What is $\det A$?

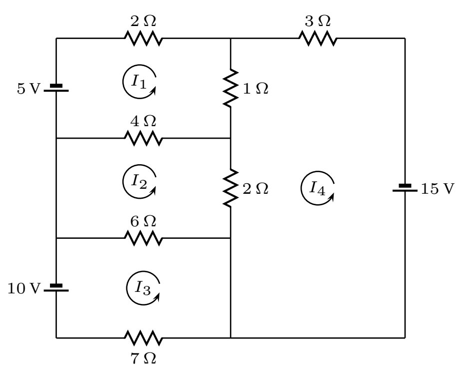
5. Consider the partitioned matrix $M = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & A \\ I & 0 & 0 \end{pmatrix}$.

a. Given that A and B are invertible, find M^{-1}.

b. Find the inverse of the matrix

$$\begin{pmatrix} 0 & 0 & 8 & 13 & 0 & 0 & 0 \\ 0 & 0 & 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

6. Write down the augmented matrix of a linear system whose solution gives the loop currents in the following electrical network.



7. Find the standard matrices of the two linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ which map the unit square onto the rectangle $\{\, s\mathbf{e}_1 + t\mathbf{e}_2 \colon -2 \leqslant s \leqslant 0,\ -1 \leqslant t \leqslant 0 \,\}$.

8. Let $\mathbf{n}$ and $\mathbf{p}$ be vectors in $\mathbb{R}^n$ with $\mathbf{n} \neq 0$, let ϖ be the hyperplane in $\mathbb{R}^n$ with normal equation $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{p}$, and let $R\colon \mathbb{R}^n \to \mathbb{R}^n$ be the mapping that reflects points in ϖ.

a. Give a formula for R using vector operations.

b. Prove that R is not a linear transformation unless $\mathbf{n}^T\mathbf{p} = 0$.

c. If $n = 4$, $\mathbf{p} = \mathbf{0}$ and

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix},$$

find the standard matrix of R. What is the kernel of R? What is the range of R?

d. Suppose that R is as in part c of this question, and let $P\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by $P(\mathbf{x}) = \frac{1}{2}\big(\mathbf{x} + R(\mathbf{x})\big)$. Find the standard matrix of P, and identify the kernel and range of P.

9. Let A and B be 4×4 matrices with B singular and $\det A = -3$. Evaluate each of the following, or indicate that there is not enough information.

a. $\det(2A)$ b. $\det(AB)$ c. $\det(A + B + I)$ d. $\det\big((A^TA)^{-1}\big)$

10. Let A, B be $n \times n$ matrices such that AB is its own inverse.

a. What is the inverse of BAB?

b. Is B necessarily invertible? Justify your answer.

c. Prove that BA is its own inverse.

d. Evaluate and simplify $(AB + I)(AB + I)$.

e. Simplify $(AB + I)^n$ as much as possible.

11. Given

$$A = \begin{pmatrix} 2 & 4 & 20 & 7 & 0 & 20 & 17 \\ 2 & -4 & -4 & -11 & -12 & -12 & -21 \\ 1 & 0 & 4 & -1 & -3 & 2 & -1 \\ -2 & 3 & 1 & 6 & 5 & -3 & 8 \end{pmatrix}$$

$$\sim R = \begin{pmatrix} 1 & 0 & 4 & 0 & -1 & 6 & 2 \\ 0 & 1 & 3 & 0 & -3 & -5 & -2 \\ 0 & 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find: a. a basis of $\operatorname{Col} A$; b. a basis of $\operatorname{Row} A$; c. a basis of $\operatorname{Nul} A$; d. $\operatorname{rank} A$; e. $\dim(\operatorname{Nul} A)$; f. $\operatorname{rank}(A^T)$; g. $\dim\big(\operatorname{Nul}(A^T)\big)$.

12. Let $\mathscr{H}$ be the set of all 2×2 matrices the sum of whose entries is zero.

a. Give an example of an invertible matrix in $\mathscr{H}$.

b. Find a basis for $\mathscr{H}$.

c. What is the dimension of $\mathscr{H}$?

13. Let $\mathscr{X} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : |x| = |y| \right\}$.

a. Is $\mathscr{X}$ closed under addition? Justify your answer.

b. Does $\mathscr{X}$ contain the zero vector of $\mathbb{R}^2$? Justify your answer.

c. Is $\mathscr{X}$ closed under scalar multiplication? Justify your answer.

d. Is $\mathscr{X}$ a subspace of $\mathbb{R}^2$? Justify your answer.

14. Give a specific example of:

a. a 2×2 matrix A such that $\operatorname{Col} A = \operatorname{Nul} A$;

b. a 3×3 matrix A with every entry different such that $\det A = 0$;

c. two mutually orthogonal vectors in $\mathbb{R}^3$ that have no zero entries;

d. a line in $\mathbb{R}^3$ that is parallel to the xy-plane;

e. an invertible 4×4 skew-symmetric matrix.

15. Given the planes

$\varpi_1\colon 2x+3y+3z = -8$, $\varpi_2\colon x+2y+2z = -6$ and $\varpi_3\colon x+2y+2z = 1$.

a. Find a parametric vector representation of the line of intersection of ϖ_1 and ϖ_2.

b. What is the cosine of the (acute) angle between ϖ_1 and ϖ_2?

c. Find the distance between ϖ_2 and ϖ_3.

16. Let ϖ be the plane containing $Q(1, 2, 3)$, $R(2, 3, 3)$ and $S(6, 4, -2)$.

a. Find a parametric vector equation of ϖ

b. Find a standard equation of the plane ϖ.

c. Find the area of the triangle QRS.

d. Find the volume of the parallelepiped formed by the vectors $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OS}$.

17. Find the point of intersection of the plane $3x - 2y + 5z = 3$ and the line

$$\begin{pmatrix} -2 \\ -4 \\ 8 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}.$$

18. a. Define (precisely) what it means for an indexed set $\{x_1, \ldots, x_p\}$ of vectors in a vector space V to be linearly independent.

b. Suppose that V, W are vector spaces, $T\colon V \to W$ is an injective linear transformation, and that $\{x_1, \ldots, x_p\}$ is linearly independent in V. Prove that $\{T(x_1), \ldots, T(x_p)\}$ is linearly independent.

1. Dividing the first equation by 2 and reducing the augmented matrix of the resulting linear system using the row reduction algorithm yields

$$\begin{pmatrix}1 & -2 & -1 & 4 & -2\\ -3 & 4 & -1 & -2 & 0\\ -1 & 3 & 3 & -9 & 5\end{pmatrix} \sim \begin{pmatrix}1 & -2 & -1 & 4 & -2\\ 0 & -2 & -4 & 10 & -6\\ 0 & 1 & 2 & -5 & 3\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & -2 & -1 & 4 & -2\\ 0 & -2 & -4 & 10 & -6\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 0 & 3 & -6 & 4\\ 0 & 1 & 2 & -5 & 3\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

Therefore, the solution set of the linear system in question consists of all vectors of the form

$$\begin{pmatrix}4\\3\\0\\0\end{pmatrix} + s\begin{pmatrix}-3\\-2\\1\\0\end{pmatrix} + t\begin{pmatrix}6\\5\\0\\1\end{pmatrix},$$

where s and t are real numbers.

2. a. Laplace expansion along column three, and then along column one gives

$$\det A = \begin{vmatrix}3 & 5 & 0 & 4\\ -1 & 3 & 1 & -2\\ 0 & k & 0 & 1\\ 0 & 4 & 0 & k\end{vmatrix} = -\begin{vmatrix}3 & 5 & 4\\ 0 & k & 1\\ 0 & 4 & k\end{vmatrix} = -3\begin{vmatrix}k & 1\\ 4 & k\end{vmatrix} = 3(4-k^2).$$

b. A is singular if, and only if, $k = \pm 2$.

3. Reducing $\begin{pmatrix}A & I_3\end{pmatrix}$ using the row reduction algorithm gives

$$\begin{pmatrix}2 & -3 & -14 & 1 & 0 & 0\\ 1 & -2 & -7 & 0 & 1 & 0\\ -3 & 5 & 22 & 0 & 0 & 1\end{pmatrix} \sim \begin{pmatrix}2 & -3 & -14 & 1 & 0 & 0\\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0\\ 0 & \frac{1}{2} & 1 & \frac{3}{2} & 0 & 1\end{pmatrix}$$
$$\sim \begin{pmatrix}2 & -3 & -14 & 1 & 0 & 0\\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\end{pmatrix}$$
$$\sim \begin{pmatrix}2 & -3 & 0 & 15 & 14 & 14\\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\end{pmatrix}$$
$$\sim \begin{pmatrix}2 & 0 & 0 & 18 & 8 & 14\\ 0 & 1 & 0 & 1 & -2 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 0 & 0 & 9 & 4 & 7\\ 0 & 1 & 0 & 1 & -2 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\end{pmatrix};$$

therefore,

$$A^{-1} = \begin{pmatrix}9 & 4 & 7\\ 1 & -2 & 0\\ 1 & 1 & 1\end{pmatrix}.$$

4. a. One has

$$A = \begin{pmatrix}3 & 2 & 6\\ 9 & 4 & 22\\ -12 & -12 & -11\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -4 & 2 & 1\end{pmatrix}\begin{pmatrix}3 & 2 & 6\\ 0 & -2 & 4\\ 0 & 0 & 5\end{pmatrix} = LU$$

via the rough work (eliding the rows and columns that go into producing L and U)

$$\begin{pmatrix}3 & 2 & 6\\ 9 & 4 & 22\\ -12 & -12 & -11\end{pmatrix} \rightsquigarrow \begin{pmatrix}-2 & 4\\ -4 & 13\end{pmatrix} \rightsquigarrow \begin{pmatrix}5\end{pmatrix}.$$

b. Since L is *unit* lower triangular, U is upper triangular, and the determinant preserves products, it follows that $\det A = \det U = (3)(-2)(5) = -30$.

5. a. The inverse of M, if it exists, is a matrix X such that $I = XM$, *i.e.*,

$$\begin{pmatrix}I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{pmatrix} = \begin{pmatrix}X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{pmatrix}\begin{pmatrix}0 & B & 0\\ 0 & 0 & A\\ I & 0 & 0\end{pmatrix}$$
$$= \begin{pmatrix}X_{13} & X_{11}B & X_{12}A\\ X_{23} & X_{21}B & X_{22}A\\ X_{33} & X_{31}B & X_{32}A\end{pmatrix}.$$

Comparing the leftmost blocks of product with those of the identity matrix gives the rightmost blocks of X, *i.e.*,

$$\begin{pmatrix}I\\0\\0\end{pmatrix} = \begin{pmatrix}X_{13}\\X_{23}\\X_{33}\end{pmatrix}.$$

Since B is invertible, comparing the middle blocks of the product with those of the identity matrix gives the leftmost blocks of X, *i.e.*,

$$\begin{pmatrix}0\\I\\0\end{pmatrix} = \begin{pmatrix}X_{11}B\\X_{21}B\\X_{31}B\end{pmatrix} = \begin{pmatrix}X_{11}\\X_{21}\\X_{31}\end{pmatrix}B, \quad \text{so} \quad \begin{pmatrix}X_{11}\\X_{21}\\X_{31}\end{pmatrix} = \begin{pmatrix}0\\B^{-1}\\0\end{pmatrix}.$$

Finally, since A is invertible, comparing the rightmost blocks of the produce with those of the identity matrix gives the middle blocks of X, *i.e.*,

$$\begin{pmatrix}0\\0\\I\end{pmatrix} = \begin{pmatrix}X_{12}A\\X_{22}A\\X_{32}A\end{pmatrix} = \begin{pmatrix}X_{12}\\X_{22}\\X_{32}\end{pmatrix}A, \quad \text{so} \quad \begin{pmatrix}X_{12}\\X_{22}\\X_{32}\end{pmatrix} = \begin{pmatrix}0\\0\\A^{-1}\end{pmatrix}.$$

Therefore,

$$\begin{pmatrix}0 & B & 0\\ 0 & 0 & A\\ I & 0 & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & 0 & I\\ B^{-1} & 0 & 0\\ 0 & A^{-1} & 0\end{pmatrix}.$$

Note: The inverse of M could be found more quickly by careful inspection.

b. Since the matrix in question has the same form as M, where

$$B = \begin{pmatrix}8 & 13\\ 3 & 5\end{pmatrix} \quad \text{so} \quad B^{-1} = \begin{pmatrix}5 & -13\\ -3 & 8\end{pmatrix},$$

and

$$A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 4 & 0 & 1\end{pmatrix} \quad \text{so} \quad A^{-1} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -4 & 0 & 1\end{pmatrix},$$

it follows from Part a that

$$\begin{pmatrix}0 & 0 & 8 & 13 & 0 & 0 & 0\\ 0 & 0 & 3 & 5 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 4 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 5 & -13 & 0 & 0 & 0 & 0 & 0\\ -3 & 8 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -4 & 0 & 1 & 0 & 0\end{pmatrix}.$$

6. Applying Ohm's Law and Kirchhoff's Law to each loop gives the equations $7I_1 - 4I_2 - I_4 = 5\,\mathrm{V}$, $-4I_1 + 12I_2 - 6I_3 - 2I_4 = 0\,\mathrm{V}$, $-6I_2 + 13I_3 = 10\,\mathrm{V}$, and $-I_1 - 2I_2 + 6I_4 = -15\,\mathrm{V}$, which linear system has augmented matrix

$$\begin{pmatrix}7 & -4 & 0 & -1 & 5\\ -4 & 12 & -6 & -2 & 0\\ 0 & -6 & 13 & 0 & 10\\ -1 & -2 & 0 & 6 & -15\end{pmatrix}.$$

7. A linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ maps the unit square onto the given rectangle if, and only if, it maps the *unordered* pair $\{\mathbf{e}_1, \mathbf{e}_2\}$ onto the *unordered* pair $\{-2\mathbf{e}^1, -\mathbf{e}_2\}$. The standard matrices of the two such linear transformations are $\begin{pmatrix}-2\mathbf{e}_1 & -\mathbf{e}_2\end{pmatrix}$ and $\begin{pmatrix}-\mathbf{e}_2 & -2\mathbf{e}_1\end{pmatrix}$, or

$$\begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0 & -2\\ -1 & 0\end{pmatrix}.$$

8. a. Let $\mathbf{y} = \mathbf{x} + 2\,\mathrm{proj}_{\mathbf{n}}(\mathbf{p}-\mathbf{x})$. Then $\mathbf{y}-\mathbf{x}$ is a scalar multiple of $\mathbf{n}$ and therefore orthogonal to ϖ. Also, the midpoint of the segment joining $\mathbf{x}$ and $\mathbf{y}$, $\frac{1}{2}(\mathbf{x}+\mathbf{y})$, lies in ϖ. Therefore $\mathbf{y}$ is the reflection of $\mathbf{x}$ in ϖ, *i.e.*,

$$R(\mathbf{x}) = \mathbf{x} + 2\,\mathrm{proj}_{\mathbf{n}}(\mathbf{p}-\mathbf{x}) = \mathbf{x} + 2\frac{\mathbf{n}^T(\mathbf{p}-\mathbf{x})}{\mathbf{n}^T\mathbf{n}}\mathbf{n}.$$

b. R is not a linear transformation unless $R(\mathbf{0}) = \mathbf{0}$, *i.e.*,

$$2\frac{\mathbf{n}^T\mathbf{p}}{\mathbf{n}^T\mathbf{n}}\mathbf{n} = \mathbf{0},$$

which implies that $\mathbf{n}^T\mathbf{p} = 0$ (since $\mathbf{n} \neq \mathbf{0}$).

c. Since $\mathbf{p} = \mathbf{0}$, R is given by

$$R(\mathbf{x}) = \mathbf{x} - 2\frac{\mathbf{n}^T\mathbf{x}}{\mathbf{n}^T\mathbf{n}}\mathbf{n},$$

and the standard matrix of R is

$$[R] = \begin{pmatrix} R(\mathbf{e}_1) & R(\mathbf{e}_2) & R(\mathbf{e}_3) & R(\mathbf{e}_4) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ -\frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix}.$$

Since R reflects points in ϖ, R is its own inverse. Therefore, the kernel of R is $\{\mathbf{0}\}$ and the range of R is $\mathbb{R}^3$.

d. By the solution to part a of this question, P is the orthogonal projection onto ϖ (of part c). Therefore, the kernel of P consists of all multiples of $\mathbf{n}$, and the range of P is ϖ. The standard matrix of P is $\frac{1}{2}(I_4 + A)$, where A is the standard matrix of R from part c; *i.e.*,

$$[P] = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} & 0 & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

9. a. Since A is a 4×4 matrix, $\det(2A) = 2^4 \det A = 16(-3) = -48$.

b. Since the determinant preserves products and B is singular, $\det(AB) = (\det A)(\det B) = (-3)(0) = 0$.

c. There is not enough information to determine the value of $\det(A+B+I)$. For example, if $B = 0$ then the value is -16 if $A = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & -3\mathbf{e}_4 \end{pmatrix}$, and the value is 0 if $A = \begin{pmatrix} -\mathbf{e}_1 & -\mathbf{e}_2 & -\mathbf{e}_3 & 3\mathbf{e}_4 \end{pmatrix}$.

d. Since the determinant preserves products and is invariant under transposition, $\det\big((A^TA)^{-1}\big) = \big((\det A^T)(\det A)\big)^{-1} = (\det A)^{-2} = (-3)^{-2} = \frac{1}{9}$.

10. a. Since $ABAB = I$, $(BAB)^{-1} = A$ by the (first corollary to the) Invertible Matrix Theorem.

b. Yes, B is invertible. For essentially the same reason as in Part a, $B^{-1} = ABA$.

c. Since $BABA = A^{-1}(ABAB)A = A^{-1}A = I$, BA is its own inverse by the (first corollary to the) Invertible Matrix Theorem.

d. $(AB+I)^2 = (AB)^2 + 2(AB) + I = 2(AB+I)$, since $(AB)^2 = I$.

e. By Part c and the induction hypothesis that $(AB+I)^{k-1} = 2^{k-2}(AB+I)$, where $k > 1$, it follows that

$$\begin{aligned}(AB+I)^k &= (AB+I)^{k-1}(AB+I) \\ &= 2^{k-2}(AB+I)^2 \\ &= 2^{k-2} \cdot 2(AB+I) \\ &= 2^{k-1}(AB+I).\end{aligned}$$

Therefore, $(AB+I)^n = 2^{n-1}(AB+I)$ for all $n \geqslant 1$ by (the principle of mathematical) induction.

11. Let $\mathbf{a}_j$ denote column j of A (for $j = 1, \ldots, 7$), and let $\mathbf{r}_i$ denote column i of R^T (for $j = 1, \ldots, 7$).

a. The list of pivot columns of A, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$, is a basis for $\operatorname{Col} A$.

b. The list of non-zero columns of R^T, $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$, is a basis for $\operatorname{Row} A$.

c. The reduced echelon form R of A reveals that the non-pivot columns of A are given by $\mathbf{a}_3 = 4\mathbf{a}_1 + 3\mathbf{a}_2$, $\mathbf{a}_5 = -\mathbf{a}_1 - 3\mathbf{a}_2 + 2\mathbf{a}_4$, $\mathbf{a}_6 = 6\mathbf{a}_1 - 5\mathbf{a}_2 + 4\mathbf{a}_4$ and $\mathbf{a}_7 = 2\mathbf{a}_1 - 2\mathbf{a}_2 + 3\mathbf{a}_4$. Therefore,

$$\left\{ \begin{pmatrix} -4 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 5 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\operatorname{Nul} A$.

d. The rank of A is three.

e. The nullity of A is is four.

f. The rank of A^T is three.

g. The nullity of A^T is 1 (*e.g.*, by the rank formula).

12. a. $\begin{pmatrix} \mathbf{e}_1 & -\mathbf{e}_2 \end{pmatrix}$ is an invertible matrix that belongs to $\mathscr{H}$.

b. A matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belongs to $\mathscr{H}$ if, and only if, $a + b + c + d = 0$, in which b, c and d are free and $a = -b - c - d$. Therefore, $\mathscr{H}$ consists of all matrices of the form

$$\begin{pmatrix} -b-c-d & b \\ c & d \end{pmatrix} = bM_1 + cM_2 + dM_3$$

where

$$M_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and b, c and d are real numbers. Since $\{M_1, M_2, M_3\}$ is linearly independent, it is a basis for $\mathscr{H}$.

c. The dimension of $\mathscr{H}$ is 3.

13. a. $\mathscr{X}$ is not closed under addition, for

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathscr{X}, \quad \text{but} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \notin \mathscr{X}.$$

b. $\mathscr{X}$ does contain the zero vector, since $|0| = |0|$.

c. $\mathscr{X}$ is closed under scalar multiplication, for if

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{X}$$

and $\alpha \in \mathbb{R}$, then $|x| = |y|$, and so $|\alpha x| = |\alpha||x| = |\alpha||y| = |\alpha y|$, which implies that

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} \in \mathscr{X}.$$

d. $\mathscr{X}$ is not a subspace of $\mathbb{R}^2$ since it is not closed under addition.

14. a. If $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, then $\operatorname{Col} A = \operatorname{Nul} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

b. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then $\mathbf{a}_1 + \mathbf{a}_3 = 2\mathbf{a}_2$, and so $\det A = 0$.

c. The vectors

$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

have no zero entries and are orthogonal (since $\mathbf{u}^T\mathbf{v} = 0$).

d. The line ℓ with parametric representation $\mathbf{e}_3 + t\mathbf{e}_1$ is parallel to the xy-plane (since the third entry of every point on ℓ is equal to 1).

e. The matrix $U = \begin{pmatrix} \mathbf{e}_2 & -\mathbf{e}_1 \end{pmatrix}$ is skew-symmetric ($U^T + U = 0$) and orthogonal ($U^TU = I$), therefore the partitioned matrix

$$A = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

is also skew-symmetric and orthogonal (so in particular A is also invertible).

15. a. Reducing the augmented matrix of the system of the given linear equations for ϖ_1 and ϖ_2 yields

$$\begin{pmatrix} 2 & 3 & 3 & -8 \\ 1 & 2 & 2 & -6 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 3 & -8 \\ 0 & \frac{1}{2} & \frac{1}{2} & -2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -4 \end{pmatrix}.$$

Therefore, on the line of intersection of ϖ_1 and ϖ_2, $z = t$ is free, $x = 2$ and $y = -4 - t$, so this line consists of all vectors of the form

$$\begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

where t is a real number.

b. The cosine of the acute angle between ϖ_1 and ϖ_2 is equal to the cosine of the acute angle between the lines spanned by their given normals, *i.e.*

$$\frac{|\mathbf{n}_1^T \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{14}{\sqrt{22}\sqrt{9}} = \frac{7}{33}\sqrt{66}, \text{ where } \mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \text{ and } \mathbf{n}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

c. The distance between the parallel planes ϖ_2 and ϖ_3 is equal to the distance between ϖ_2 and the x-intercept $\mathbf{e}_1$ of ϖ_3 (or any point on ϖ_3 for that matter), which distance is given by

$$\frac{|(1) + 2(0) + 2(0) - (-6)|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{7}{3}.$$

16. In this problem, let

$$\mathbf{p} = \overrightarrow{OQ} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{u} = \overrightarrow{QR} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \overrightarrow{QS} = \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}.$$

a. The plane ϖ consists of all vectors of the form $\mathbf{p} + s\mathbf{u} + t\mathbf{v}$, where s and t are real numbers.

$$\mathbf{n} = -\mathbf{u} \times \mathbf{v} = -\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix},$$

and a standard equation of ϖ is obtained by expanding the normal equation $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\overrightarrow{OQ}$, which yields $5x - 5y + 3z = 4$.

b. The area of the triangle QRS is equal to

$$\tfrac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \tfrac{1}{2}\sqrt{5^2 + (-5)^2 + 3^2} = \tfrac{1}{2}\sqrt{59}.$$

c. The volume of the parallelepiped formed by $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OS}$ is equal to

$$\left|\det\begin{pmatrix} \overrightarrow{OQ} & \overrightarrow{OR} & \overrightarrow{OS} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & 2 & 6 \\ 2 & 3 & 4 \\ 3 & 3 & -2 \end{pmatrix}\right| = \left|\det\begin{pmatrix} -1 & -8 \\ -3 & -20 \end{pmatrix}\right| = 4.$$

17. The line meets the plane where its entries satisfy the given equation of the plane. This happens if $3(-2+2t) - 2(-4+2t) + 5(8-3t) = 3$, *i.e.*, $13t = 39$, or $t = 3$. Therefore, the point of intersection the line and the plane is

$$\begin{pmatrix} -2 \\ -4 \\ 8 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}.$$

18. a. An indexed set $\{x_1, \ldots, x_p\}$ in a vector space V is linearly independent if, whenever $\alpha_1, \ldots, \alpha_p$ are scalars such that $\alpha_1 x_1 + \cdots + \alpha_p x_p = 0_V$ (the zero vector of V), then $\alpha_j = 0$ for $j = 1, \ldots, p$.

b. Suppose $\alpha_1, \ldots, \alpha_p$ are scalars such that

$$\alpha_1 T(x_1) + \cdots + \alpha_p T(x_p) = 0_W$$

(the zero vector of W). Since T is a linear transformation, this last equation implies that

$$T(\alpha_1 x_1 + \cdots \alpha_p x_p) = T(0_V),$$

and since T is injective it follows that

$$\alpha_1 x_1 + \cdots \alpha_p x_p = 0_V.$$

But $\{x_1, \ldots, x_p\}$ is linearly independent, so $\alpha_j = 0$ for $j = 1, \ldots, p$. This shows that the equation $\alpha_1 T(x_1) + \cdots + \alpha_p T(x_p) = 0_W$ implies that $\alpha_j = 0$ for $j = 1, \ldots, p$; *i.e.*, that $\{T(x_1), \ldots, T(x_p)\}$ is linearly independent.