1. Given

$$A = \begin{pmatrix} 1 & 1 & -1 & 8 \\ -4 & -3 & 1 & -26 \\ -5 & -3 & 1 & -30 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}.$$

a. Solve the equation $A\mathbf{x} = \mathbf{b}$. Express your answer in parametric vector form.
b. Find a particular solution of $A\mathbf{x} = \mathbf{b}$ whose first entry is 6.
c. Give a basis of $\operatorname{Nul} A$.

2. Given the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & b \\ 0 & 3 & a & b \end{pmatrix}.$$

Under what conditions on a and b is $\operatorname{rank} A$ equal to a. 4? b. 3? c. 2?

3. Let

$$A = \begin{pmatrix} 3 & -2 & -4 \\ 1 & -1 & -3 \\ 0 & 4 & 21 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix}.$$

a. Find A^{-1}.
b. Solve $A\mathbf{x} = \mathbf{b}$ using your answer to Part a.

4. Assume that the partitioned matrix

$$M = \begin{pmatrix} 0 & B & 0 \\ A & C & 0 \\ 0 & 0 & D \end{pmatrix}$$

is invertible, and that A, B, C and D are square.

a. Express M^{-1} as a partitioned matrix.
b. Which of the matrices A, B, C, D must be invertible for M^{-1} to exist?

5. Set up an augmented matrix for balancing the chemical equation

$$\mathrm{Ca(OH)_2} + \mathrm{H_3PO_4} \to \mathrm{Ca_3(PO_4)_2} + \mathrm{H_2O}$$

6. Let A be a 4×4 symmetric matrix with $\det A = -5$. For each part, either provide an answer or write "not enough information".

a. What is the value of $\det(-4A^{-1})$?
b. What is the value of $\det(2A^T - A)$?
c. What is the value of $\det(A - I)$?

7. Let A, B and C be $n \times n$ matrices and suppose that $AB^TC^{-1} = I_n$.

a. Use determinants to explain why A and B must be invertible.
b. Does A commute with B^TC^{-1}? Explain your answer.
c. Find B^{-1}.

8. Express

$$A = \begin{pmatrix} 2 & 5 \\ -2 & -8 \\ 8 & 2 \end{pmatrix}$$

as a product LU, where L is lower triangular and U is upper triangular.

9. Find elementary matrices E_1 and E_2 such that

$$E_2E_1 \begin{pmatrix} -5 & 6 \\ 0 & 1 \end{pmatrix} = I_2.$$

10. Let $\mathscr{P}_1$ be the parallelogram with vertices $(0,0)$, $(1,-2)$, $(4,-1)$ and $(3,1)$, let $\mathscr{P}_2$ be the parallelogram with vertices $(0,0)$, $(-1,-1)$, $(1,-4)$ and $(2,-3)$, and let A be a 2×2 matrix, such that the linear transformation $\mathbf{x} \rightsquigarrow A\mathbf{x}$ maps $\mathscr{P}_1$ onto $\mathscr{P}_2$.
a. What are the possible values of $\det A$?
b. Give a specific matrix B such that the linear transformation $\mathbf{x} \rightsquigarrow B\mathbf{x}$ maps $\mathscr{P}_2$ onto $\mathscr{P}_1$.

11. Simplify the expression

$$\mathbf{u} \cdot \big((\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})\big)$$

as much as possible, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$.

12. Let

$$\mathscr{H} = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : ad = bc \right\} \subset \mathbb{R}^4.$$

a. Does $\mathscr{H}$ contain the zero vector of $\mathbb{R}^4$?
b. Is $\mathscr{H}$ closed under addition? Justify your answer.
c. Is $\mathscr{H}$ closed under scalar multiplication? Justify your answer.
d. Is $\mathscr{H}$ a subspace of $\mathbb{R}^4$? Justify your answer.

13. Find a specific example of each of the following, if possible. If there is no such example, explain why not.

a. A non-zero 2×2 matrix A such that $\operatorname{Col} A = \operatorname{Row} A$.
b. A 2×2 matrix A such that $\operatorname{Nul} A = \operatorname{Row} A$.
c. A lower triangular 3×3 matrix A such that A and $A + I$ are both singular.
d. A square matrix A such that $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is surjective but not injective.
e. A unit vector that is orthogonal to both

$$\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

14. You are given points $A(2,7,-1)$, $B(3,3,-1)$, $C(3,7,-4)$ and $D(5,5,5)$.

a. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
b. Find $\operatorname{proj}_{\overrightarrow{AC}} \overrightarrow{AB}$ and $\operatorname{perp}_{\overrightarrow{AC}} \overrightarrow{AB}$.
c. Find the distance from the point B to the line AC.
d. Find a normal equation of the plane containing the points A, B and C.
e. Find the volume of the parallelepiped with edges $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

15. Let ϖ_1 and ϖ_2 be, respectively the planes with standard equations

$$4x - 2y + 5z = 3 \quad \text{and} \quad -2x + y + kz = 0.$$

a. For which values of k, if any, are ϖ_1 and ϖ_2 parallel?
b. For which values of k, if any, are ϖ_1 and ϖ_2 orthogonal?
c. For which values of k, if any, does the point $(-1,-1,1)$ lie on ϖ_1 and ϖ_2?

16. Let A and B be matrices of the same size. Suppose that $\mathbf{x}$ is in both $\operatorname{Nul} A$ and $\operatorname{Nul} B$. Show that $\mathbf{x}$ must be in $\operatorname{Nul}(A + B)$.

17. Find a basis of the subspace

$$\mathscr{H} = \{\, X \in M_{2\times 2} \colon AX = XA \,\}$$

of $M_{2\times 2}$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

18. Let $U = \operatorname{Span}\{\mathbf{p}, \mathbf{q}\}$ and $V = \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 1 \end{pmatrix}.$$

a. Find a basis of $U + V$.
b. Find a basis of $U \cap V$.
c. What is the distance between the line $\mathbf{p} + t\mathbf{q}$ and V?
d. Find the point(s) on U which minimize the distance to the line $\mathbf{u} + t\mathbf{v}$, and find the distance between U and this line.

19. a. Give the (precise) definition of a basis of a vector space.
b. Suppose that V and W are vector spaces, that $\mathscr{B} = \{b_1, \ldots, b_n\}$ is a basis of V, and that $T \colon V \to W$ is a linear transformation such that

$$T(b_i) = 0_W \quad \text{for } i = 1, \ldots, n.$$

Prove that T is the zero transformation.
c. Suppose that V is a vector space, and that

$$\mathscr{A} = \{a_1, \ldots a_m\} \quad \text{and} \quad \mathscr{B} = \{b_1, \ldots b_n\}$$

are bases of V. Prove that $m = n$.

1. a. Reducing the augmented matrix of the given equation yields

$$\begin{pmatrix}1&1&-1&8&-4\\-4&-3&1&-26&0\\-5&-3&1&-30&2\end{pmatrix}\sim\begin{pmatrix}1&1&-1&8&-4\\0&1&-3&6&-16\\0&2&-4&10&-18\end{pmatrix}$$
$$\sim\begin{pmatrix}1&1&-1&8&-4\\0&1&-3&6&-16\\0&0&2&-2&14\end{pmatrix}$$
$$\sim\begin{pmatrix}1&1&0&7&3\\0&1&0&3&5\\0&0&1&-1&7\end{pmatrix}$$
$$\sim\begin{pmatrix}1&0&0&4&-2\\0&1&0&3&5\\0&0&1&-1&7\end{pmatrix}.$$

Therefore, the general solution of $A\mathbf{x}=\mathbf{b}$ consists of all vectors of the form $\mathbf{p}+t\mathbf{u}$, where t is a real number,

$$\mathbf{p}=\begin{pmatrix}-2\\5\\7\\0\end{pmatrix}\quad\text{and}\quad\mathbf{u}=\begin{pmatrix}-4\\-3\\1\\1\end{pmatrix}.$$

b. $\mathbf{p}-2\mathbf{u}$, *i.e.*,

$$\begin{pmatrix}-2\\5\\7\\0\end{pmatrix}-2\begin{pmatrix}-4\\-3\\1\\1\end{pmatrix}=\begin{pmatrix}6\\11\\5\\-2\end{pmatrix}$$

is a solution of $A\mathbf{x}=\mathbf{b}$ whose first entry is 6.

c. The null space of A, *i.e.*, the solution set of $A\mathbf{x}=\mathbf{0}$, consists of all vectors of the form $t\mathbf{u}$, where t is a real number. So $\{\mathbf{u}\}$ is a basis of $\operatorname{Nul} A$

2. Reducing the matrix A gives

$$\begin{pmatrix}1&0&2&1\\1&1&1&2\\1&2&0&b\\0&3&a&b\end{pmatrix}\sim\begin{pmatrix}1&0&2&1\\0&1&-1&1\\0&2&-2&b-1\\0&3&a&b\end{pmatrix}$$
$$\sim\begin{pmatrix}1&0&2&1\\0&1&-1&1\\0&0&0&b-3\\0&0&a+3&b-3\end{pmatrix}$$
$$\sim\begin{pmatrix}1&0&2&1\\0&1&-1&1\\0&0&a+3&b-3\\0&0&0&b-3\end{pmatrix}.$$

a. The rank of A is equal to 4 if $a\neq-3$ and $b\neq3$.

b. The rank of A is equal to 3 if $a=-3$ and $b\neq3$, or $a\neq-3$ and $b=3$.

c. The rank of A is equal to 2 if $a=-3$ and $b=3$.

3. a. Reducing $\begin{pmatrix}A & I_3\end{pmatrix}$ gives

$$\begin{pmatrix}3&-2&-4&1&0&0\\1&-1&-3&0&1&0\\0&4&21&0&0&1\end{pmatrix}\sim\begin{pmatrix}3&-2&-4&1&0&0\\0&-\frac{1}{3}&-\frac{5}{3}&-\frac{1}{3}&1&0\\0&4&21&0&0&1\end{pmatrix}$$
$$\sim\begin{pmatrix}3&-2&-4&1&0&0\\0&-\frac{1}{3}&-\frac{5}{3}&-\frac{1}{3}&1&0\\0&0&1&-4&12&1\end{pmatrix}$$
$$\sim\begin{pmatrix}3&-2&0&-15&48&4\\0&-\frac{1}{3}&0&-7&21&\frac{5}{3}\\0&0&1&-4&12&1\end{pmatrix}$$
$$\sim\begin{pmatrix}3&0&0&27&-78&-6\\0&1&0&21&-63&-5\\0&0&1&-4&12&1\end{pmatrix}$$
$$\sim\begin{pmatrix}1&0&0&9&-26&-2\\0&1&0&21&-63&-5\\0&0&1&-4&12&1\end{pmatrix},$$

and therefore

$$A^{-1}=\begin{pmatrix}9&-26&-2\\21&-63&-5\\-4&12&1\end{pmatrix}.$$

b. Since A is invertible, the unique solution to $A\mathbf{x}=\mathbf{b}$ is

$$\mathbf{x}=A^{-1}\mathbf{b}=\begin{pmatrix}9&-26&-2\\21&-63&-5\\-4&12&1\end{pmatrix}\begin{pmatrix}3\\1\\8\end{pmatrix}=\begin{pmatrix}-15\\-40\\8\end{pmatrix}.$$

4. a. Since the inverse of a nonsingular block diagonal matrix is block diagonal, the inverse of M, if it exists, has the form

$$N=\begin{pmatrix}P&Q&0\\R&S&0\\0&0&T\end{pmatrix},$$

with square blocks. Comparing the blocks of NM,

$$\begin{pmatrix}P&Q&0\\R&S&0\\0&0&T\end{pmatrix}\begin{pmatrix}0&B&0\\A&C&0\\0&0&D\end{pmatrix}=\begin{pmatrix}QA&PB+QC&0\\SA&RB+SC&0\\0&0&TD\end{pmatrix},$$

to those of the block diagonal identity matrix, gives

$$QA=I,\qquad PB+QC=0,$$
$$SA=0,\qquad RB+SC=I$$

and $TD=I$. The equations $QA=I$ and $TD=I$, since the factors on the left sides are square, give $Q=A^{-1}$ and $T=D^{-1}$. Then $SA=0$ implies that $S=0$. Since $S=0$, the equation $RB+SC=I$ becomes $RB=I$ and, again since the factors on the left side are square, it follows that $R=B^{-1}$. Finally, the equation $PB+QC=0$ becomes $PB+A^{-1}C=0$, or $PB=-A^{-1}C$, *i.e.*, $P=-A^{-1}CB^{-1}$. Therefore, the inverse of M, if it exists, is

$$\begin{pmatrix}-A^{-1}CB^{-1}&A^{-1}&0\\B^{-1}&0&0\\0&0&D^{-1}\end{pmatrix}.$$

b. If M is invertible then A, B and D are invertible.

5. If (taking Calcium, Oxygen, Hydrogen and Phosphorus, in order)

$$\mathbf{s}_1=\begin{pmatrix}1\\2\\2\\0\end{pmatrix},\quad\mathbf{s}_2=\begin{pmatrix}0\\4\\3\\1\end{pmatrix},\quad\mathbf{s}_3=\begin{pmatrix}3\\8\\0\\2\end{pmatrix}\quad\text{and}\quad\mathbf{s}_4=\begin{pmatrix}0\\1\\2\\0\end{pmatrix}$$

then positive integer solutions of the vector equation $x_1\mathbf{s}_1+x_2\mathbf{s}_2=x_3\mathbf{s}_3+x_4\mathbf{s}_4$ balance the chemical equation in question. The corresponding augmented matrix is $\begin{pmatrix}\mathbf{s}_1&\mathbf{s}_2&-\mathbf{s}_3&-\mathbf{s}_4&\mathbf{0}\end{pmatrix}$.

6. Using that $\det A = -5$ and $A^T = A$, one finds that
a. $\det(-4A^{-1}) = (-4)^4(\det A)^{-1} = -\frac{256}{5}$ and
b. $\det(2A^T - A) = \det A = -5$, but that
c. $\det(A - I)$ is not determined, for example if $A = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & -5\mathbf{e}_4\end{pmatrix}$ then $\det(A-I) = 0$, and if $A = \begin{pmatrix}2\mathbf{e}_1 & 2\mathbf{e}_2 & 2\mathbf{e}_3 & -\frac{5}{8}\mathbf{e}_4\end{pmatrix}$ then $\det(A-I) = -\frac{13}{8}$.

7. a. If $I_n = AB^TC^{-1}$ then, since the determinant preserves products and is invariant under transposition, $(\det A)(\det B)(\det C)^{-1} = 1$, which implies that $\det A \neq 0$ and $\det B \neq 0$, so A and B are nonsingular.
b. Since A and B^TC^{-1} are $n \times n$ matrices and $AB^TC^{-1} = I_n$, it follows that $A^{-1} = B^TC^{-1}$, which implies that A commutes with B^TC^{-1}.
c. By Part b, $B^TC^{-1}A = I_n$, which implies (since B^T and $C^{-1}A$ are $n \times n$ matrices) that $(B^T)^{-1} = C^{-1}A$, and therefore $B^{-1} = (C^{-1}A)^T$.

8. Here is an LU factorization of A

$$\begin{pmatrix} 2 & 5 \\ -2 & -8 \\ 8 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 4 & 6 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 0 & -3 \\ 0 & 0 \end{pmatrix},$$

via the rough work

$$\boxed{\begin{matrix} -3 \\ -18 \end{matrix}} \rightsquigarrow \boxed{0}$$

9. By inspection, one has

$$\begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -5 & 6 \\ 0 & 1 \end{pmatrix}.$$

10. The parallelogram $\mathscr{P}_1$ is formed by $\mathbf{u}_1$ and $\mathbf{u}_2$, and $\mathscr{P}_2$ is the parallelogram formed by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

a. The area of the parallelograms $\mathscr{P}_1$ and $\mathscr{P}_2$ are, respectively,

$$\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7 \quad \text{and} \quad \begin{vmatrix} -1 & 2 \\ -1 & -3 \end{vmatrix} = 5,$$

so the determinant of a linear transformation which maps $\mathscr{P}_1$ onto $\mathscr{P}_2$ is equal to $\pm\frac{5}{7}$ (depending on whether or not the transformation preserves or reverses orientation).
b. One such matrix B will satisfy $B\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}$, *i.e.*,

$$\begin{aligned} B &= \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix}^{-1} \\ &= \tfrac{1}{5}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 1 & -1 \end{pmatrix} \\ &= \tfrac{1}{5}\begin{pmatrix} 0 & -5 \\ 7 & 3 \end{pmatrix}. \end{aligned}$$

The other such matrix is

$$\begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}\mathbf{v}_2 & \mathbf{v}_1\end{pmatrix}^{-1} = -\tfrac{1}{5}\begin{pmatrix} 8 & 7 \\ 5 & 0 \end{pmatrix}.$$

11. The bilinearity of the dot and cross product, the algebraic characterization of the cross product and the alternating property of the determinant, gives

$$\begin{aligned} &\mathbf{u}\cdot((\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u})) \\ &\quad = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w} - \mathbf{v}\times\mathbf{u} - \mathbf{u}\times\mathbf{w} + \mathbf{u}\times\mathbf{u}) \\ &\quad = \det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} - \det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{u}\end{pmatrix} - \det\begin{pmatrix}\mathbf{u} & \mathbf{u} & \mathbf{w}\end{pmatrix} \\ &\quad = \det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}. \end{aligned}$$

12. a. Since $0 \cdot 0 = 0 \cdot 0$, the zero vector of $\mathbb{R}^4$ belongs to $\mathscr{H}$.
b. $\mathbf{e}_2$ and $\mathbf{e}_3$ belong to $\mathscr{H}$, since each side of the equation defining $\mathscr{H}$ is zero, but $\mathbf{e}_2 + \mathbf{e}_3$ does not belong to $\mathscr{H}$, because the left side of the equation defining $\mathscr{H}$ is zero and the right side is 1. Therefore, $\mathscr{H}$ is not closed under addition.
c. If $\mathbf{x}$ belongs to $\mathscr{H}$ and α is a scalar, then $x_1x_4 = x_2x_3$, and therefore $(\alpha x_1)(\alpha x_4) = \alpha^2 x_1x_4 = \alpha^2 x_2x_3 = (\alpha x_2)(\alpha x_3)$, so $\alpha\mathbf{x}$ belongs to $\mathscr{H}$. This shows that $\mathscr{H}$ is closed under scalar multiplication.
d. Since $\mathscr{H}$ is not closed under addition, it is not a subspace of $\mathbb{R}^4$.

13. a. If A is any symmetric matrix then $\operatorname{Col} A = \operatorname{Row} A$. Simple examples such matrices are I_2, $\begin{pmatrix}\mathbf{e}_1 & \mathbf{0}\end{pmatrix}$, $\begin{pmatrix}\mathbf{0} & \mathbf{e}_2\end{pmatrix}$, $\begin{pmatrix}\mathbf{e}_2 & \mathbf{e}_1\end{pmatrix}$, $\begin{pmatrix}\mathbf{e}_1+\mathbf{e}_2 & \mathbf{e}_1+\mathbf{e}_2\end{pmatrix}$, & *c*.
b. If $\mathbf{y} \in \operatorname{Nul} A$ and $\mathbf{y} \in \operatorname{Row} A = \operatorname{Col} A^T$ then $\mathbf{y} = A^T\mathbf{x}$ for some $\mathbf{x}$ and so $\mathbf{y}^T\mathbf{y} = \mathbf{y}^TA^T\mathbf{x} = (A\mathbf{y})^T\mathbf{x} = \mathbf{0}^T\mathbf{x} = 0$, which implies that $\mathbf{y} = \mathbf{0}$. Hence, if $\operatorname{Nul} A \subset \operatorname{Row} A$ then $\operatorname{Nul} A = \{\mathbf{0}\}$, which implies that A is invertible and therefore $\operatorname{Row} A = \mathbb{R}^2$. This shows that there is no 2×2 matrix A such that $\operatorname{Nul} A = \operatorname{Row} A$.
c. If $A = \begin{pmatrix}\mathbf{0} & \mathbf{0} & -\mathbf{e}_3\end{pmatrix}$ then $\det A = 0$, and $A + I_3 = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0}\end{pmatrix}$, so $\det(A + I_3) = 0$, as required.
d. If $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is surjective then A is invertible (by the Invertible Matrix Theorem), which implies that $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is injective. So there is no matrix as requested.
e. The cross product

$$\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix},$$

is orthogonal to each of the given vectors, so its normalization,

$$\tfrac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},$$

is a vector as required

14. In this problem one has

$$\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \overrightarrow{AD} = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}.$$

a. The cosine of the angle between $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$ is equal to

$$\frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{\sqrt{10}\sqrt{17}} = \tfrac{1}{170}\sqrt{170}.$$

b. The orthogonal projection of $\mathbf{u} = \overrightarrow{AB}$ onto $\mathbf{v} = \overrightarrow{AC}$ is equal to

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{u}_{\mathbf{v}} = \frac{\mathbf{v}^T\mathbf{u}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \tfrac{1}{10}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$$

and the orthogonal component of the projection is

$$\operatorname{perp}_{\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathbf{u}_{\mathbf{v}} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix} - \tfrac{1}{10}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \tfrac{1}{10}\begin{pmatrix} 9 \\ -40 \\ 3 \end{pmatrix}.$$

c. The distance from the point B to the line AC is equal to the length of the orthogonal component of the projection of $\mathbf{u} = \overrightarrow{AB}$ onto $\mathbf{v} = \overrightarrow{AC}$, *i.e.*,

$$\|\operatorname{perp}_{\mathbf{v}}\mathbf{u}\| = \tfrac{1}{10}\sqrt{1690} = \tfrac{13}{10}\sqrt{10}.$$

Alternatively, the distance between the point B and the line AC is equal to the area of the parallelogram formed by $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$, divided by the length of (the base) $\mathbf{v} = \overrightarrow{AC}$. The area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ is the is the length of the cross product

$$\mathbf{n} = \mathbf{u}\times\mathbf{v} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix},$$

i.e., 13, and the length of $\mathbf{v}$ is $\sqrt{10}$ which, as expected, gives a distance of $\frac{13}{10}\sqrt{10}$.
d. The vector $\mathbf{n}$ (from Part c) is a normal to the plane containing the points A, B and C, so a normal equation of this plane is $\mathbf{n}^T{=}\mathbf{n}^T\mathbf{p}$, *i.e.*,

$$12x_1 + 3x_2 + 4x_3 = 41$$

where for example $\mathbf{p}$ is $\overrightarrow{OA}$, $\overrightarrow{OB}$ or $\overrightarrow{OC}$.
e. The volume of the parallelepiped with edges $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{AC}$, $\mathbf{w} = \overrightarrow{AD}$ is equal to, 54, *i.e.*, (the absolute value of) $\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} = (\mathbf{u}\times\mathbf{v})^T\mathbf{w}$.

15. Let $\mathbf{n}_1$ be the given normal to the plane ϖ_1, and let $\mathbf{n}_2$ be the given normal to the plane ϖ_2.

a. The planes ϖ_1 and ϖ_2 are parallel if their normal vectors are so; *i.e.*, if $\mathbf{n}_2 = -\frac{1}{2}\mathbf{n}_1$, or $k = -\frac{5}{2}$.

b. The planes ϖ_1 and ϖ_2 are orthogonal if their normal vectors are so; *i.e.*, if $0 = \mathbf{n}_1^T\mathbf{n}_2 = -10 + 5k$, or $k = 2$.

c. The point $P(-1,-1,1)$ lies on the planes ϖ_1 and ϖ_2 if its coordinates satisfy the equations of both planes. The point P is on the plane ϖ_1 since $4(-1) - 2(-1) + 5(1) = 3$, and it is also on ϖ_2 if $-2(-1) + (-1) + k(1) = 0$, *i.e.*, $k = -1$.

16. If $\mathbf{x}$ is in both $\operatorname{Nul} A$ and $\operatorname{Nul} B$, then $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, which shows that $\mathbf{x}$ belongs to $\operatorname{Nul}(A+B)$.

17. If

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

then

$$AX = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}+x_{21} & x_{12}+x_{22} \\ -x_{21} & -x_{22} \end{pmatrix}$$

and

$$XA = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{11}-x_{12} \\ x_{21} & x_{21}-x_{22} \end{pmatrix},$$

and so by comparing entries one finds that $AX = XA$ if, and only if, $x_{21} = 0$ and $x_{11} = 2x_{12} + x_{22}$. Therefore, $X \in \mathscr{H}$ if, and only if,

$$X = \begin{pmatrix} x_{22}+2x_{12} & x_{12} \\ 0 & x_{22} \end{pmatrix} = x_{22}M_1 + x_{12}M_2,$$

where

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad\text{and}\quad M_2 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $\{M_1, M_2\}$ spans $\mathscr{H}$, and is clearly linearly independent, it is a basis of $\mathscr{H}$.

18. a. Since $U + V = \operatorname{Span}\{\mathbf{p},\mathbf{q},\mathbf{u},\mathbf{v}\}$, and since

$$\begin{aligned}
\begin{pmatrix}\mathbf{p} & \mathbf{q} & \mathbf{u} & \mathbf{v}\end{pmatrix} &= \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 3 \\ 1 & 2 & 1 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

from which it follows that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$ are linearly independent and $\mathbf{v} = 3\mathbf{p} - \mathbf{q}$, the list $\{\mathbf{p},\mathbf{q},\mathbf{u}\}$ is a basis of $U + V$.

b. From the solution to Part a it follows that every vector in $U \cap V$ is a multiple of $\mathbf{v} = 3\mathbf{p} - \mathbf{q}$. Therefore, $\{\mathbf{v}\}$ is a basis of $U \cap V$.

c. By the solution to Part a, $\frac{1}{3}\mathbf{v} = \mathbf{p} - \frac{1}{3}\mathbf{q}$ belongs to V and to the line $\mathbf{p} + t\mathbf{q}$, from which it follows that the distance between V and this line is zero.

d. Let $A = \begin{pmatrix}\mathbf{p} & \mathbf{q}\end{pmatrix}$; then, since $\mathbf{v}$ is parallel to U, the points on U which minimize the distance to the line $\mathbf{u} + t\mathbf{v}$ are of the form $\mathbf{x} + t\mathbf{v}$, where

$$\begin{aligned}
\mathbf{x} &= A(A^TA)^{-1}A^T\mathbf{u} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \tfrac{1}{3}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \tfrac{1}{3}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \tfrac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}
\end{aligned}$$

is the orthogonal projection of $\mathbf{u}$ onto U. The distance between U and the line $\mathbf{u} + t\mathbf{v}$ is equal to the length of

$$\mathbf{u} - \mathbf{x} = \tfrac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 2 \\ 0 \end{pmatrix},$$

which is $\frac{2}{3}\sqrt{3}$.

19. a. An indexed set $\mathscr{B}$ of vectors in a linear space V is a basis of V if $\mathscr{B}$ is linearly independent and $\operatorname{Span}\mathscr{B} = V$.

b. If $x \in V$ then there are scalars $\beta_1, \ldots, \beta_n$ such that $x = \beta_1 b_1 + \cdots + \beta_n b_n$, and so

$$\begin{aligned}
T(x) &= T(\beta_1 b_1 + \cdots + \beta_n b_n) \\
&= \beta_1 T(b_1) + \cdots + \beta_n T(b_n) \\
&= 0_W,
\end{aligned}$$

since $T(b_j) = 0_W$ for $j = 1, \ldots, n$., and so T is the zero transformation.

c. Since $\operatorname{Span}\mathscr{A} = V$ (because $\mathscr{A}$ is a basis of V), there is an $n \times m$ matrix $A = \begin{pmatrix}\alpha_{ji}\end{pmatrix}$ such that

$$b_j = \alpha_{j1}a_1 + \cdots + \alpha_{jm}a_m = \sum_{i=1}^{m} \alpha_{ji}a_i$$

for $j = 1, \ldots, n$. Likewise, there is an $m \times n$ matrix $B = \begin{pmatrix}\beta_{ij}\end{pmatrix}$ such that

$$a_i = \beta_{i1}b_1 + \cdots + \beta_{in}b_n = \sum_{j=1}^{n} \beta_{ij}b_j$$

for $i = 1, \ldots, n$ (since $\mathscr{B}$ is a basis of V and hence $\operatorname{Span}\mathscr{B} = V$). So for $k = 1, \ldots, n$,

$$\begin{aligned}
b_k &= \sum_{i=1}^{m} \alpha_{ki}a_i = \sum_{i=1}^{m} \alpha_{ki}\left(\sum_{j=1}^{n} \beta_{ij}b_j\right) \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} \alpha_{ki}\beta_{ij}\right)b_j,
\end{aligned}$$

which implies that

$$\sum_{i=1}^{m} \alpha_{ki}\beta_{ij} = \begin{cases} 1 & \text{if } k = j\text{, and} \\ 0 & \text{if } k \neq j, \end{cases}$$

i.e., $AB = I_n$, since $\mathscr{B}$ is linearly independent. By a similar computation, the fact that $\mathscr{A}$ is linearly independent implies that $BA = I_m$. It follows that $m = n$ and that A and B are inverses (*e.g.*, because $AB = I_n$ implies that B has linearly independent columns, so $n \leqslant m$, and likewise $BA = I_m$ implies that A has linearly independent columns, so $m \leqslant n$).

Note: A more compact solution is available if you know about coordinate vectors and the list-vector product (the product of a list of abstract vectors and a column vector): Since $\operatorname{Span}\mathscr{A} = V$ there is a $m \times n$ matrix P such that $\mathscr{A}P = \mathscr{B}$. If $P\mathbf{x} = \mathbf{0}$ then $\mathscr{B}\mathbf{x} = \mathscr{A}P\mathbf{x} = \mathbf{0}$, and so, since $\mathscr{B}$ is linearly independent, $\mathbf{x} = \mathbf{0}$. Hence, P has linearly independent columns, which implies that $n \leqslant m$. Likewise, since $\operatorname{Span}\mathscr{B} = V$ and $\mathscr{A}$ is linearly independent, it follows that $m \leqslant n$.