**1.** Solve the linear system

$$\begin{aligned} x_1 + \phantom{2}x_2 - \phantom{4}x_3 - 2x_4 + \phantom{5}x_5 &= 1 \\ 2x_1 + \phantom{2}x_2 + \phantom{4}x_3 + 2x_4 - \phantom{5}x_5 &= 2 \\ x_1 + 2x_2 - 4x_3 - 8x_4 + 5x_5 &= 1 \\ x_2 - 3x_3 - 6x_4 + 3x_5 &= 0. \end{aligned}$$

**2.** Let

$$A = \begin{pmatrix} 2 & 6 & -5 \\ -1 & -3 & 3 \\ 1 & 4 & -6 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 8 \\ -4 \\ 5 \end{pmatrix}.$$

a. Find $A^{-1}$.
b. Use your answer to part a to solve the equation $A\mathbf{x} = \mathbf{b}$.

**3.** Let

$$\mathbf{u}_1 = \begin{pmatrix} x \\ x \\ 2 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} x \\ 2 \\ x \end{pmatrix} \quad \text{and} \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ x \\ -x \end{pmatrix}.$$

a. For which values of $x$, if any, is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ linearly dependent?
b. For which values of $x$, if any, is $\{\mathbf{u}_1, \mathbf{u}_2\}$ linearly dependent?
c. For which values of $x$, if any, is $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ equal to $\mathbb{R}^3$?
d. For which values of $x$, if any, is $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ a line in $\mathbb{R}^3$?

**4.** For each of the following, give an example or explain why no there is no such example.
a. A $2 \times 3$ matrix such that the linear transformation $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is injective.
b. A $2 \times 3$ matrix each of whose entries is $\pm 1$, such that the linear transformation $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is not surjective.
c. A matrix $A$ such that $A^2$ is invertible but $A$ is not invertible.
d. A non-zero $2 \times 2$ matrix $A$ such that $A^2 = 0$.

**5.** Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates points about the origin by $\frac{1}{4}\pi$ (radians in the counterclockwise direction), and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects points in the line $x + y = 0$.
a. Give the standard matrices of $T_1$ and $T_2$.
b. Give the standard matrix of $T_1 \circ T_2$.
c. Let $\mathscr{S} = \{\, s\mathbf{e}_1 + t\mathbf{e}_2 : 0 \leqslant s, t \leqslant 1 \,\}$ be the unit square in $\mathbb{R}^2$. Draw pictures of $\mathscr{S}$ and $(T_1 \circ T_2)(\mathscr{S})$.

**6.** Let

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}.$$

a. For which values of $\alpha$, if any, does $\begin{pmatrix} 3 \\ \alpha \end{pmatrix}$ belong to i. $\operatorname{Col} A$? ii. $\operatorname{Nul} A$?
b. Give a basis of $\operatorname{Nul} A^2$.
c. Is $\operatorname{Nul} A = \operatorname{Nul} A^2$? Justify your answer

**7.** Suppose that $A$ and $B$ are $n \times n$ matrices, $A$ has linearly independent columns, and $B$ is nonsingular.
a. Simplify $(BAB^{-1})^2$.
b. Simplify $(BAB^{-1})^{-1}$.
c. Does $BAB^{-1}$ have linearly independent columns? Justify your answer.

**8.** Fill in the blanks. The missing word is *must*, *might* or *cannot*.
a. If $\mathbf{y} \in \operatorname{Col} A$ then $A\mathbf{x} = \mathbf{y}$ _____ be inconsistent.
b. If $\mathbf{y} \in \operatorname{Col} A$ then $\mathbf{y}$ _____ be in $\operatorname{Nul} A$.
c. If $\mathbf{y} \in \operatorname{Col} A$ then $\mathbf{y}$ _____ be in $\operatorname{Row} A^T$.
d. If $\mathbf{x} \in \operatorname{Col} A$ and $\mathbf{y} \in \operatorname{Col} A$ then $\mathbf{x} + \mathbf{y}$ _____ belong to $\operatorname{Col} A$.
e. If $A$ is a $5 \times 7$ matrix then $\operatorname{Row} A$ and $\operatorname{Col} A$ _____ have the same dimension.
f. If $A$ is a $5 \times 7$ matrix then $\operatorname{Nul} A$ _____ be 3 dimensional.
g. If $A$ is a $5 \times 7$ matrix of rank 4 then $\operatorname{Nul} A^T$ _____ be 3 dimensional.

**9.** Let

$$A = \begin{pmatrix} 2 & -3 & 4 \\ 8 & -8 & 18 \\ 6 & -17 & 13 \end{pmatrix}.$$

a. Find a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $A = LU$.
b. Do the same for $A^T$.
c. What is $\det A$?
d. Find an elementary matrix $E$ such that

$$EA = \begin{pmatrix} 2 & -3 & 4 \\ 8 & -8 & 18 \\ 0 & -8 & 1 \end{pmatrix}.$$

**10.** Let $U$ be an $n \times n$ matrix which is partitioned as

$$U = \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}.$$

a. Assume that $A$ is invertible. Write $U^{-1}$ as a partitioned matrix.
b. Use your answer to part a to find the inverse of

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 7 & 5 & 3 & 2 & 6 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}.$$

**11.** Let

$$A = \begin{pmatrix} 2 & 3 & 3 & 2 \\ 4 & 3 & 5 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

a. Find $\det A$.
b. Use Cramer's Rule to solve the equation $A\mathbf{x} = \mathbf{b}$ for $x_4$ only.
c. What is $\det(-2A^{-1})$.
d. What is $\det(A^{-1}A^TA)$?

**12.** Let

$$\mathscr{V} = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a = 2d \text{ and } bc \leqslant 0 \right\}.$$

a. Is $\mathbf{0}$ in $\mathscr{V}$?
b. Is $\mathscr{V}$ closed under scalar multiplication? Justify your answer. No credit is given without an adequate justification.
c. Is $\mathscr{V}$ closed under addition? Justify your answer. No credit is given without an adequate justification.
d. Is $\mathscr{V}$ a subspace of $\mathbb{R}^4$?

**13.** Let $V = \{\, p \in \mathbb{P}_2 : p'(1) = p(1) \text{ and } p'(2) = p(2) \,\}$. Given that $V$ is a subspace of $\mathbb{P}_2$, find a basis for $V$ and state the dimension of $V$.

**14.** Let

$$\mathbf{u} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} a \\ -2 \\ b \end{pmatrix}.$$

Find
a. the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$;
b. the area of the triangle determined by $\mathbf{u}$ and $\mathbf{v}$;
c. the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$;
d. all values of $a$ and $b$ such that $\mathbf{x}$ is orthogonal to $\mathbf{u}$.

**15.** Given the plane $\varpi$ with standard equation $x + y - z = 11$, and the line $\ell$ consisting of all vectors of the form

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

where $t$ is a real number, find
a. the point of intersection of $\varpi$ and $\ell$;
b. the distance from the point $Q(2, -1, 3)$ to the plane $\varpi$;
c. the distance from the point $R(1, 0, 1)$ to the line $\ell$.

**16.** a. Show that the lines

$$\ell_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \qquad \ell_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

intersect in a point, and find the point of intersection.
b. Find a standard equation of the plane containing $\ell_1$ and $\ell_2$.

**17.** a. Show that if $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\|$ then $\mathbf{x}^T\mathbf{y} \leqslant 0$.
b. Prove the identity $\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 = 4\mathbf{x}^T\mathbf{y}$.

**18.** Let $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$ be an $n \times n$ matrix such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
a. Show that each column of $A$ is a unit vector.
b. Show that any two different columns of $A$ are mutually orthogonal.
c. Show that $A^TA = I_n$.

**19.** A matrix $X$ is called a *weak generalized inverse* of $A$ if $AXA = A$.
a. For which values of $\xi$, if any, is

$$\begin{pmatrix} \xi & \xi \\ \xi & \xi \\ \xi & \xi \end{pmatrix} \text{ a weak generalized inverse of } \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}?$$

For parts b and c suppose that $A$ is an $m \times n$ matrix and that $X$ is a weak generalized inverse of $A$.
b. Show that if $\mathbf{y}$ is any vector in $\mathbb{R}^n$ then $(I_n - XA)\mathbf{y}$ belongs to $\operatorname{Nul} A$.
c. Show that if the system $A\mathbf{x} = \mathbf{b}$ is consistent then $X\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$.
d. Prove that every $m \times n$ matrix $A$ has a weak generalized inverse.

**1.** Reducing the augmented matrix gives

$$\begin{pmatrix} 1 & 1 & -1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 2 & -1 & 2 \\ 1 & 2 & -4 & -8 & 5 & 1 \\ 0 & 1 & -3 & -6 & 3 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & -1 & -2 & 1 & 1 \\ 0 & -1 & 3 & 6 & -3 & 0 \\ 0 & 1 & -3 & -6 & 4 & 0 \\ 0 & 1 & -3 & -6 & 3 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & -1 & -2 & 1 & 1 \\ 0 & -1 & 3 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 0 & 1 & -3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and so the general solution of the linear system in question is the set of all vectors of the form

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 6 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where $s$ and $t$ are real numbers.

**2.** a. Reducing $\begin{pmatrix} A & I_3 \end{pmatrix}$ gives

$$\begin{pmatrix} A & I_3 \end{pmatrix} \sim \begin{pmatrix} 2 & 6 & -5 & 1 & 0 & 0 \\ -1 & -3 & 3 & 0 & 1 & 0 \\ 1 & 4 & -6 & 0 & 0 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & 6 & -5 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{7}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & 6 & -5 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & 6 & 0 & 6 & 10 & 0 \\ 0 & 1 & 0 & 3 & 7 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & 0 & 0 & -12 & -32 & -6 \\ 0 & 1 & 0 & 3 & 7 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & -6 & -16 & -3 \\ 0 & 1 & 0 & 3 & 7 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{pmatrix};$$

therefore,

$$A^{-1} = \begin{pmatrix} -6 & -16 & -3 \\ 3 & 7 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

b. Using the result of Part a,

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -6 & -16 & -3 \\ 3 & 7 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ -4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

**3.** a. Laplace expansion along the first row of $\det \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$ gives

$$\begin{aligned} \det \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix} &= \begin{vmatrix} x & x & 1 \\ x & 2 & x \\ 2 & x & -x \end{vmatrix} \\ &= x \begin{vmatrix} 2 & x \\ x & -x \end{vmatrix} - x \begin{vmatrix} x & x \\ 2 & -x \end{vmatrix} + \begin{vmatrix} x & 2 \\ 2 & x \end{vmatrix} \\ &= -x^2(x+2) + x^2(x+2) + x^2 - 4 \\ &= x^2 - 4, \end{aligned}$$

so $\{\mathbf{u}_1, \mathbf{u}_2\mathbf{u}_3\}$ is linearly dependent if, and only if, $x = \pm 2$.

b. $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly dependent if, and only if, there is a scalar $\alpha$ such that $\alpha\mathbf{u}_1 = \mathbf{u}_2$, or, $\alpha x = x$, $\alpha x = 2$ and $2\alpha = x$, *i.e.*, $\alpha = 2$.

c. Since the dimension of $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is at most 2, it is never equal to $\mathbb{R}^3$.

d. $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is a line in $\mathbb{R}^3$ precisely when its dimension is one; *i.e.*, when $x = 2$.

**4.** a. If $A$ is a $2 \times 3$ matrix then $\dim \operatorname{Nul} A \geqslant 1$, so there is no such example.

b. Any such $2 \times 3$ matrix of rank one will do; for example, if

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix},$$

then $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is not surjective (its range is the line $x_1 + x_2 = 0$).

c. If $A^2$ is (defined and) invertible, then $A$ is square and $A^{-1} = A(A^2)^{-1}$, so no such example is possible.

d. Any non-zero $2 \times 2$ matrix $A$ for which $\operatorname{Col} A \subset \operatorname{Nul} A$ will do; for example, if $A = \begin{pmatrix} \mathbf{e}_2 & \mathbf{0} \end{pmatrix}$ then $A^2$ is the zero $2 \times 2$ matrix.

**5.** a. The standard matrix of $T_1$ is

$$A = \begin{pmatrix} \cos \frac{1}{4}\pi & -\sin \frac{1}{4}\pi \\ \sin \frac{1}{4}\pi & \cos \frac{1}{4}\pi \end{pmatrix} = \tfrac{1}{2}\sqrt{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and the standard matrix of $T_2$ is

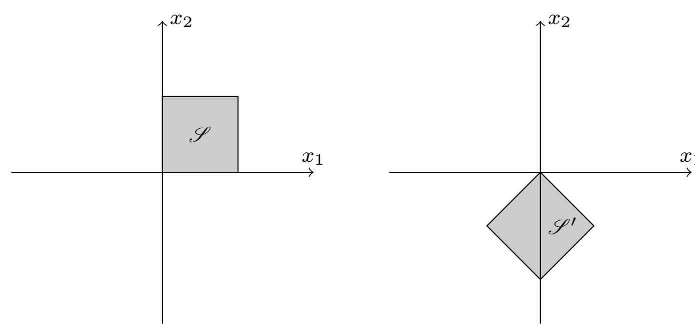
$$B = \begin{pmatrix} -\mathbf{e}_2 & -\mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

since $T_2(\mathbf{e}_1) = -\mathbf{e}_2$ and $T_2(\mathbf{e}_2) = -\mathbf{e}_1$.

b. The standard matrix of $T_1 \circ T_2$ is

$$AB = -\tfrac{1}{2}\sqrt{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

c. Below are pictures of $\mathscr{S}$ and $\mathscr{S}' = (T_1 \circ T_2)(\mathscr{S})$.




**6.** Observe that $A = \begin{pmatrix} \mathbf{u} & -2\mathbf{u} \end{pmatrix}$, where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a. The vector

$$\begin{pmatrix} 3 \\ \alpha \end{pmatrix}$$

belongs to: i. $\operatorname{Col} A$ if, and only if, $\alpha = 2 \cdot 3 = 6$; ii. $\operatorname{Nul} A$ if, and only if $3 - 2\alpha = 0$, *i.e.*, $\alpha = \frac{3}{2}$.

b. Observe that $A^2 = \begin{pmatrix} -3\mathbf{u} & 6\mathbf{u} \end{pmatrix} = -3A$, so

$$\mathscr{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

is a basis of $\operatorname{Nul} A^2$.

c. Since $A^2 = -3A$, it follows that $\operatorname{Nul} A = \operatorname{Nul} A^2$.

**7.** Observe that since $A$ is a square matrix with linearly independent columns, $A$ is nonsingular.
a. $(BAB^{-1})^2 = BAB^{-1}BAB^{-1} = BA^2B^{-1}$.
b. $(BAB^{-1})^{-1} = (B^{-1})^{-1}A^{-1}B^{-1} = BA^{-1}B^{-1}$.
c. $BAB^{-1}$ is invertible, so its columns are linearly independent.

**8.** a. If $\mathbf{y} \in \operatorname{Col} A$ then $A\mathbf{x} = \mathbf{y}$ *cannot* be inconsistent.
b. If $\mathbf{y} \in \operatorname{Col} A$ then $\mathbf{y}$ *might* be in $\operatorname{Nul} A$.
c. If $\mathbf{y} \in \operatorname{Col} A$ then $\mathbf{y}$ *must* be in $\operatorname{Row} A^T$.
d. If $\mathbf{x} \in \operatorname{Col} A$ and $\mathbf{y} \in \operatorname{Col} A$ then $\mathbf{x}+\mathbf{y}$ *must* belong to $\operatorname{Col} A$.
e. If $A$ is a $5 \times 7$ matrix then $\operatorname{Row} A$ and $\operatorname{Col} A$ *must* have the same dimension.
f. If $A$ is a $5 \times 7$ matrix then $\operatorname{Nul} A$ *might* be 3 dimensional.
g. If $A$ is a $5 \times 7$ matrix of rank 4 then $\operatorname{Nul} A^T$ *cannot* be 3 dimensional.

**9.** a. Here is an $LU$ factorization of $A$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix},$$

via the rough work

$$A \rightsquigarrow \boxed{\begin{matrix} 4 & 2 \\ -8 & 1 \end{matrix}} \rightsquigarrow \boxed{5}.$$

b. Using Part a, one finds that $A^T$ factorizes as $U^TL^T$ (in the question it was not required that the left factor be unit lower triangular); *i.e.*,

$$\begin{pmatrix} 2 & 0 & 0 \\ -3 & 4 & 0 \\ 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

c. Using the factorization from Part a, one has $\det A = \det U = 2 \cdot 4 \cdot 5 = 40$.
d. The given product $EA$ is the result of adding $-3$ times the first row of $A$ to the third row of $A$, so the elementary matrix $E$ is given by

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}.$$

**10.** a. The equation

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 & I \\ A & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

is (by comparing blocks) equivalent to

$$\begin{aligned} QA &= I, & P + QB &= 0, \\ SA &= 0 \quad \text{and} & R + SB &= I. \end{aligned}$$

Since $A$ is invertible, the equations on the left are equivalent to $Q = A^{-1}$ and $S = 0$, which gives $R = I$ and $P = -A^{-1}B$. Therefore,

$$U^{-1} = \begin{pmatrix} -A^{-1}B & A^{-1} \\ I & 0 \end{pmatrix}.$$

b. The matrix in question can be partitioned as in Part a so that

$$A = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 & 6 \\ 2 & 1 & 5 \end{pmatrix},$$

and hence

$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix} \quad \text{and} \quad -A^{-1}B = \begin{pmatrix} 1 & -1 & 7 \\ -2 & 1 & -11 \end{pmatrix},$$

which gives

$$M^{-1} = \begin{pmatrix} 1 & -1 & 7 & 3 & -5 \\ -2 & 1 & -11 & -4 & 7 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

**11.** a. One has

$$\det A = -\begin{vmatrix} 3 & 3 & 2 & 2 \\ 5 & 3 & 4 & 1 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 7 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 3 \\ 5 & 3 \end{vmatrix} \cdot \begin{vmatrix} 6 & 3 \\ 7 & 4 \end{vmatrix} = -(-6)(3) = 18,$$

since the determinant of a block triangular matrix with square diagonal blocks is the product of the determinants of its diagonal blocks.
b. Since

$$\det A_4(\mathbf{b}) = -\begin{vmatrix} 3 & 3 & 2 & 1 \\ 5 & 3 & 4 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 7 & 0 \end{vmatrix} = -\begin{vmatrix} 3 & 3 \\ 5 & 3 \end{vmatrix} \cdot \begin{vmatrix} 6 & 1 \\ 7 & 0 \end{vmatrix} = -(-6)(-7) = -42,$$

it follows that $x_4 = -\frac{7}{3}$.
c. Since $A$ is a $4 \times 4$ matrix,

$$\det\bigl(-2A^{-1}\bigr) = (-2)^4(\det A)^{-1} = \tfrac{8}{9}.$$

d. Since the determinant preserves products and is invariant under transposition,

$$\det\bigl(A^{-1}A^TA\bigr) = (\det A)^{-1}(\det A)^2 = 18.$$

**12.** a. Since $0 = 2 \cdot 0$ and $0 \cdot 0 \leqslant 0$, the zero vector of $R^4$ does belong to $\mathscr{V}$.
b. If $\mathbf{x} \in \mathscr{V}$ and $\alpha$ is any scalar, then $x_1 = 2x_4$ and $x_2x_3 \leqslant 0$, from which it follows that $(\alpha x_1) = 2(\alpha x_4)$ and $(\alpha x_2)(\alpha x_3) = \alpha^2 x_2 x_3 \leqslant 0$ (since $\alpha^2 \geqslant 0$). Therefore, $\alpha\mathbf{x} \in \mathscr{V}$. This shows that $\mathscr{V}$ is closed under scalar multiplication.
c. Observe that

$$\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

each belong to $\mathscr{V}$, but their sum

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

does not belong to $\mathscr{V}$ (since the product of its second and third entries is positive). This shows that $\mathscr{V}$ is not closed under addition.
d. Since $\mathscr{V}$ is not closed under addition, $\mathscr{V}$ is not a subspace of $\mathbb{R}^4$.

**13.** If $p(x) = a_0 + a_1x + a_2x^2$ belongs to $V$ then, since $p'(x) = a_1 + 2a_2x$, it follows that

$$a_1 + 2a_2 = a_0 + a_1 + a_2, \qquad \text{or} \qquad a_2 = a_0,$$

since $p'(1) = p(1)$, and

$$a_1 + 4a_2 = a_0 + 2a_1 + 4a_2, \qquad \text{or} \qquad a_1 = -a_0,$$

since $p'(2) = p(2)$. Therefore, $p(x) = a_0 - a_0x + a_0x^2 = a_0(1 - x + x^2)$, which shows that $\{1 - x + x^2\}$ (since it spans $V$ and is linearly independent) is a basis of $V$. The dimension of $V$ is one.

**14.** a. The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ is equal to

$$\frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{18}{\sqrt{14}\sqrt{25}} = \tfrac{9}{35}\sqrt{14}.$$

b. The area of the triangle determined by $\mathbf{u}$ and $\mathbf{v}$ is equal to half the length of the cross product

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix},$$

which is $\frac{1}{2}\sqrt{26}$.
c. The volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is equal to the absolute value of the determinant

$$\det\bigl(\mathbf{u}\ \mathbf{v}\ \mathbf{w}\bigr) = (\mathbf{u} \times \mathbf{v})^T\mathbf{w},$$

which is 7.

d. $\mathbf{x}$ is orthogonal to $\mathbf{u}$ if, and only if,

$$\mathbf{u}^T\mathbf{x} = 3a + 2b + 2$$

is equal to zero. An explicit description is given by all vectors of the form

$$\begin{pmatrix}-2\\0\end{pmatrix} + t\begin{pmatrix}-2\\3\end{pmatrix},$$

where $t$ is a real number.

**15.** a. The line $\ell$ meets the plane $\varpi$ where the coordinates of a point on the line satisfy the equation of the plane; *i.e.*, where $(1+t)+(2t)-(-2t) = 11$, or $5t = 10$, so $t = 2$ and therefore the line meets the plane at

$$\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}3\\4\\-4\end{pmatrix}.$$

b. The distance between the point $Q$ and the plane $\varpi$ is equal to

$$\frac{|\mathbf{n}^T\overrightarrow{OQ} - 11|}{\|\mathbf{n}\|} = \frac{|-2-11|}{\sqrt{3}} = \tfrac{13}{3}\sqrt{3},$$

where

$$\overrightarrow{OQ} = \begin{pmatrix}2\\-1\\3\end{pmatrix} \qquad \text{and} \qquad \mathbf{n} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

are, respectively, the position vector of $Q$ and a normal vector to the plane $\varpi$.

c. The distance between the point $R$ and the line $\ell$ is equal to the area of the parallelogram formed by $\overrightarrow{OR} - \mathbf{e}_1 = \mathbf{e}_3$ and the given direction vector $\mathbf{u}$, divided by the length of $\mathbf{u}$. The area of the parallelogram is the length of the cross product

$$\mathbf{e}_3 \times \mathbf{u} = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix};$$

*i.e.*, $\sqrt{5}$, and the length of $\mathbf{u}$ is 3. Therefore, the distance between the point $R$ and the line $\ell$ is equal to $\frac{1}{3}\sqrt{5}$.

**16.** a. The lines $\ell_1$ and $\ell_2$ intersect where

$$\begin{pmatrix}3\\4\\1\end{pmatrix} + s\begin{pmatrix}4\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\7\\5\end{pmatrix} + t\begin{pmatrix}4\\2\\1\end{pmatrix}, \quad \text{or} \quad s\begin{pmatrix}4\\1\\0\end{pmatrix} - t\begin{pmatrix}4\\2\\1\end{pmatrix} = \begin{pmatrix}-4\\3\\4\end{pmatrix},$$

and so the reduction

$$\begin{pmatrix}4 & 4 & -4\\1 & 2 & 3\\0 & 1 & 4\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -5\\0 & 1 & 4\\0 & 0 & 0\end{pmatrix}$$

gives $s = -5$ and $-t = 4$, *i.e.*, $t = -4$. Using the given parametric representation of either line, one finds that the lines intersect at

$$\begin{pmatrix}-17\\-1\\1\end{pmatrix}.$$

b. A normal to the plane containing the lines $\ell_1$ and $\ell_2$ is given by the cross product

$$\mathbf{n} = \begin{pmatrix}4\\1\\0\end{pmatrix} \times \begin{pmatrix}4\\2\\1\end{pmatrix} = \begin{pmatrix}1\\-4\\4\end{pmatrix}$$

of their direction vectors, and so an equation of this plane is $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{p}$, *i.e.*, $x_1 - 4x_2 + 4x_3 = -9$, where $\mathbf{p}$ is any point on the plane (*e.g.*, either of the given points on the lines, or the point of intersection found in Part a).

**17.** a. If $\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\|$ then, since

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y}) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\mathbf{x}^T\mathbf{y}$$

one has $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\mathbf{x}^T\mathbf{y} \leqslant \|\mathbf{x}\|^2$, and so $\mathbf{x}^T\mathbf{y} \leqslant -\frac{1}{2}\|\mathbf{y}\|^2 \leqslant 0$.

b. Simplifying the left side of the inequality in question gives

$$\begin{aligned}&(\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y}) - (\mathbf{x}-\mathbf{y})^T(\mathbf{x}-\mathbf{y})\\ &\qquad = \mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{y} + 2\mathbf{x}^T\mathbf{y} - (\mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{y} - 2\mathbf{x}^T\mathbf{y})\\ &\qquad = 4\mathbf{x}^T\mathbf{y},\end{aligned}$$

as required.

**18.** a. For $j = 1, \ldots, n$, $\|\mathbf{a}_j\| = \|A\mathbf{e}_j\| = \|\mathbf{e}_j\| = 1$, which shows that each column of $A$ is a unit vector.

b. If $i \neq j$ then

$$\|\mathbf{a}_i + \mathbf{a}_j\|^2 = \|\mathbf{a}_i\|^2 + \|\mathbf{a}_j\|^2 + 2\mathbf{a}_i^T\mathbf{a}_j = 2 + 2\mathbf{a}_i^T\mathbf{a}_j$$

by Part a, and

$$\|\mathbf{a}_i + \mathbf{a}_j\|^2 = \|A(\mathbf{e}_i + \mathbf{e}_j)\|^2 = \|\mathbf{e}_i + \mathbf{e}_j\|^2 = 2,$$

which implies that $\mathbf{a}_i^T\mathbf{a}_j = 0$; *i.e.*, $\mathbf{a}_i$ and $\mathbf{a}_j$ are mutually orthogonal.

c. Since $A^TA = \mathbf{a}_i^T\mathbf{a}_j$, Parts a and b imply that

$$(A^TA)_{ij} = \begin{cases}1 & \text{if } i = j, \text{ and}\\ 0 & \text{if } i \neq j;\end{cases}$$

which shows that $A^TA = I_n$.

**19.** a. Since

$$\begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}\xi & \xi\\ \xi & \xi\\ \xi & \xi\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\end{pmatrix} = 6\xi\begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\end{pmatrix},$$

it follows that

$$\begin{pmatrix}\xi & \xi\\ \xi & \xi\\ \xi & \xi\end{pmatrix} \quad \text{is a weak generalized inverse of} \quad \begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\end{pmatrix}$$

if, and only if, $6\xi = 1$, *i.e.*, $\xi = \frac{1}{6}$.

b. If $X$ is a weak generalized inverse of $A$ then $A(I_n - XA) = A - AXA = 0$, which implies that $A(I_n - XA)\mathbf{y} = \mathbf{0}$, *i.e.*, $(I_n - XA)\mathbf{y} \in \operatorname{Nul} A$, for all $\mathbf{y} \in \mathbb{R}^n$.

c. If $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$ then $AX\mathbf{b} = AXA\mathbf{p} = A\mathbf{p} = \mathbf{b}$, which shows that $X\mathbf{b}$ is also a solution of $A\mathbf{x} = \mathbf{b}$.

d. Let $A$ be an $m \times n$ matrix of rank $r$. If $r = 0$ then $A$ is the $m \times n$ zero matrix, and the $n \times m$ zero matrix is a weak generalized inverse of $A$. On the other hand, if $r > 1$ then $A$ factorizes as $ME$ where $M$ is an $m \times r$ matrix, $E$ is an $r \times n$ and $\operatorname{rank} M = \operatorname{rank} E = r$ (for example, take the columns of $M$ to be the pivot columns of $A$, in order, and take the rows of $E$ to be the non-zero rows of the reduced echelon form of $A$, in order). There is an $n \times r$ matrix $X$ such that $EX = I_r$ (*e.g.*, take column $j$ of $X$ to be a solution of $E\mathbf{x} = \mathbf{e}_j$, which exists because $E$ has a pivot position in every row), and there is an $r \times m$ matrix $Y$ such that $YM = I_r$ (*e.g.*, take the rows of $Y$ to be the first $r$ rows of $B$, in the same order, where $BM$ is the reduced echelon form of $M$; $YM = I_r$ because $M$ has a pivot position in every column). If $Z = XY$ then

$$AZA = (ME)(XY)(ME) = M(EX)(YM)E = MI_rI_rE = ME = A,$$

which shows that $Z$ is a weak generalized inverse of $A$.