1. Given the linear system

$$\begin{aligned} -x_1 + x_2 + 5x_3 + 8x_4 - 6x_5 &= 1 \\ - x_2 - 2x_3 - 3x_4 + 4x_5 &= -2 \\ 2x_1 \qquad\quad - 6x_3 - 10x_4 + 4x_5 &= \alpha. \end{aligned}$$

a. For which values of α, if any is the linear system consistent.

b. Find the general solution of the linear system when it is consistent.

2. Given

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 1 \\ 13 \\ 16 \end{pmatrix},$$

determine whether the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. If not, give a non-trivial dependence relation satisfied by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

3. Use matrix methods to balance the chemical equation $C_4H_{10} + O_2 \longrightarrow CO_2 + H_2O$.

4. Let

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & \lambda & 4 \\ 3 & -2 & 0 & -2 \\ -2 & 2 & 3 & 4 \end{pmatrix}.$$

a. Find the determinant of A. b. For which values of λ, if any, is $\det A = \det(A^{-1})$?

5. Suppose that $\det A = 10$, where

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

a. Find the determinant of

$$\begin{pmatrix} 2a & 2b & 2c \\ 5g & 5h & 5i \\ d & e & f \end{pmatrix}.$$

b. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the columns of A. Find $\mathbf{u}^T(\mathbf{v} \times \mathbf{w})$.

6. Show that the mapping $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ x - |y| \end{pmatrix}$$

is not a linear transformation.

7. Let

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 6 \\ -2 \end{pmatrix},$$

and let S and T be the linear transformations defined by $S(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$.

a. Is S injective? Justify your answer. b. Find a vector $\mathbf{x}$ such that $S(\mathbf{x}) = \mathbf{b}$.

c. What is the range of T? Justify your answer.

d. Find a non-zero vector in the kernel of T. e. Find the standard matrix of $S \circ T$.

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates points by $\frac{3}{4}\pi$ about the origin and then reflects vectors in the line $y = x$. Find the standard matrix of T.

9. Suppose that A, B and C are square matrices, and that $ABC = I$.

a. Prove that A, B and C are invertible. b. Find B^{-1}.

10. Let $X = D^TD + I$. Prove that $X^T = X$.

11. Find A^{-1}, where

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

12. The 2×2 matrix A can be row reduced to I_2 by the following sequence of elementary row operations.

- Interchange row 1 and row 2.
- Multiply row 2 by $\frac{1}{2}$.
- Replace row 1 with the sum of itself and -4 times row 2.
- Multiply row 1 by $\frac{1}{3}$.

a. Express A as a product of elementary matrices. b. What is $\det A$?

13. Find an LU factorization of

$$A = \begin{pmatrix} 2 & -3 & 1 & 2 \\ 4 & -4 & 5 & 3 \\ -6 & 13 & 4 & -6 \end{pmatrix}.$$

14. You are given the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

in block notation, where A_{11} is an invertible matrix.

a. Find matrices X, Y and S such that

$$A = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}.$$

b. Suppose that A is a 9×13 matrix of rank 7, and that A_{11} is a 4×4 matrix.

i. What is the rank of S?

ii. What is the dimension of the null space of S?

For each of the following statements (c–e), determine whether it is true or false. Justify your answers.

c. If $A^T = A$ then $S^T = S$.

d. If A is invertible then A_{22} is invertible.

e. If A is invertible then S is invertible.

15. Let $\mathbf{b}$ be a fixed vector in $\mathbb{R}^n$ and let $V = \{\, A \in \mathrm{M}_{m\times n} : A\mathbf{b} = \mathbf{0} \,\}$.

a. Prove that V is a subspace of $\mathrm{M}_{m\times n}$.

b. Find a basis of the linear space $V = \{\, A \in \mathrm{M}_{2\times 2} : A\mathbf{b} = \mathbf{0} \,\}$, where

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

c. Prove that if A is any matrix in the space V from Part b then A^2 is a scalar multiple of A.

16. Let

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : xy = z^2 \right\}.$$

Prove that S has the stated property or use a counterexample to show that the property fails.

a. S is closed under addition. b. S is closed under scalar multiplication.

17. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & -3 \\ 1 & 2 & 7 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \\ 1 & 0 & 3 \end{pmatrix}.$$

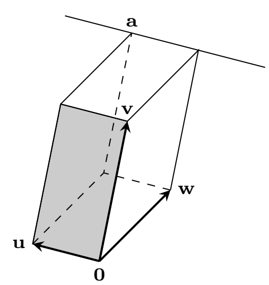
a. Find a basis of $\mathrm{Col}\, A$. b. Find a basis of $\mathrm{Nul}\, A$.

c. Find a basis of $\mathrm{Row}\, B$. d. Prove that $\mathrm{Col}\, A = \mathrm{Row}\, B$.

18. Suppose that A is an $n \times n$ matrix such that $\mathrm{Nul}\, A = \mathrm{Col}\, A$.

a. Prove that n must be even. b. Prove that $A^2 = 0$.

19. The figure below shows the parallelepiped formed by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, together with a line containing its upper back edge.



$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}.$$

a. Find the area of the shaded face.

b. Find an equation of the plane containing the shaded face.

c. Find the volume of the parallelepiped.

d. Find the coordinates of the point labelled $\mathbf{a}$ in the figure.

e. Find an equation of the line containing the upper back edge (shown in the figure).

20. Let ℓ_1 be the line $s\mathbf{e}_2$ and let ℓ_2 be the line $\mathbf{p} + t\mathbf{u}$, where

$$\mathbf{p} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

a. Find the distance between ℓ_1 and ℓ_2.

b. Find the point on ℓ_2 which is closest to the origin.

21. Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ a \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

a. Find a unit vector orthogonal to $\mathbf{u}$. b. Find $\|\mathrm{proj}_{\mathbf{v}}\, \mathbf{u}\|$. c. Find $\lim\limits_{a\to\infty} \mathrm{proj}_{\mathbf{v}}\, \mathbf{u}$.

1. Reducing the augmented matrix of the given linear system yields

$$\begin{pmatrix} -1 & 1 & 5 & 8 & -6 & 1 \\ 0 & -1 & -2 & -3 & 4 & -2 \\ 2 & 0 & -6 & -10 & 4 & \alpha \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 5 & 8 & -6 & 1 \\ 0 & -1 & -2 & -3 & 4 & -2 \\ 0 & 2 & 4 & 6 & -8 & \alpha+2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & -5 & 2 & 1 \\ 0 & 1 & 2 & 3 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & \alpha-2 \end{pmatrix};$$

therefore,

a. the linear system in question is consistent if, and only if, $\alpha = 2$, and

b. if $\alpha = 2$ then the general solution of the linear system consists of all vectors of the form $\mathbf{p} + r\mathbf{u} + s\mathbf{v} + t\mathbf{w}$, where r, s and t are real numbers,

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 5 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -2 \\ 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

2. Reducing the matrix $\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$ gives

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & -1 & 13 \\ 2 & 3 & 16 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

from which it follows that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, and that $\mathbf{w} = 5\mathbf{u} + 2\mathbf{v}$, or $-5\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$, is a non-trivial dependence relation satisfied by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

3. Positive integer solutions of the matrix equation $x_1\mathbf{s}_1 + x_2\mathbf{s}_2 = x_3\mathbf{s}_3 + x_4\mathbf{s}_4$, where

$$\mathbf{s}_1 = \begin{pmatrix} 4 \\ 10 \\ 0 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{s}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{s}_4 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

and the entries in each vector represent, in order the Carbon, Hydrogen and Oxygen atoms in the each species, balance the chemical equation in question. Reducing the matrix $\begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & -\mathbf{s}_3 & -\mathbf{s}_4 \end{pmatrix}$ gives

$$\begin{pmatrix} 4 & 0 & -1 & 0 \\ 10 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & \frac{5}{2} & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{13}{10} \\ 0 & 0 & 1 & -\frac{4}{5} \end{pmatrix},$$

which implies that positive integer solutions of the foregoing vector equation are of the form $k\mathbf{u}$, where k is a positive integer and

$$\mathbf{u} = \begin{pmatrix} 2 \\ 13 \\ 8 \\ 10 \end{pmatrix}.$$

Therefore, the chemical equation in question balances as

$$2\,C_4H_{10} + 13\,O_2 \longrightarrow 8\,CO_2 + 10\,H_2O.$$

4. a. Applying replacement elementary row operations to A gives

$$\det A = \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & \lambda & 4 \\ 3 & -2 & 0 & -2 \\ -2 & 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & \lambda & 4 \\ 0 & 0 & 2\lambda-9 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2\lambda - 9.$$

b. $\det A = \det(A^{-1})$ if, and only if, $(\det A)^2 = 1$, or $\det A = \pm 1$; *i.e.*, $2\lambda - 9 = \pm 1$, or $\lambda = 4, 5$.

5. a. The alternating multilinearity of the determinant implies that the determinant in question is equal to $(-1)(2)(5)\det A = -100$.

b. Depending on the order of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in A, $\mathbf{u}^T(\mathbf{v} \times \mathbf{w}) = \det\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$ is either $\det A = 10$ or $-\det A = -10$.

6. Since

$$T(\mathbf{e}_2) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad T(-\mathbf{e}_2) = \begin{pmatrix} -2 \\ -1 \end{pmatrix},$$

$T(-\mathbf{e}_2) \neq -T(\mathbf{e}_2)$, and so T does not preserve scalar multiplication. Therefore, T is not a linear transformation.

7. a. Since $\det A = 1 \neq 0$, S is invertible and therefore injective.

b. The unit such vector is

$$A^{-1}\begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 36 \\ -22 \end{pmatrix}.$$

c. Since

$$B \sim \begin{pmatrix} 1 & 0 & -18 \\ 0 & 1 & 11 \end{pmatrix},$$

it follows that T is surjective, and so the range of T is $\mathbb{R}^2$.

d. From the reduction in Part c, it follows that the kernel of T is $\operatorname{Span}\{\mathbf{u}\}$, where

$$\mathbf{u} = \begin{pmatrix} 18 \\ -11 \\ 1 \end{pmatrix};$$

so any non-zero multiple of $\mathbf{u}$ (*e.g.*, $\mathbf{u}$ itself) is a non-zero vector in the kernel of T.

e. The standard matrix of $S \circ T$ is the product of the standard matrices of S and T, *i.e.*,

$$AB = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 19 & 11 \\ 18 & 31 & 17 \end{pmatrix}.$$

8. The standard matrix of a $\frac{3}{4}\pi$ rotation about the origin is

$$B = \begin{pmatrix} \cos\frac{3}{4}\pi & -\sin\frac{3}{4}\pi \\ \sin\frac{3}{4}\pi & \cos\frac{3}{4}\pi \end{pmatrix} = -\tfrac{1}{2}\sqrt{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

and the standard matrix of reflection in the line $y = x$ is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so the standard matrix of their composite (rotate, then reflect) is

$$AB = -\tfrac{1}{2}\sqrt{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = -\tfrac{1}{2}\sqrt{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

9. a. Since $1 = \det I = \det(ABC) = (\det A)(\det B)(\det C)$, it follows that each of A, B, and C have non-zero determinants, and are therefore invertible.

b. Since $ABC = I$ implies that $BCA = I$ (because A and BC are square, and therefore inverses by the Invertible Matrix Theorem), it follows (by the same reasoning) that and so $B^{-1} = CA$.

10. By a direct calculation, one has

$$\begin{aligned} X^T &= \left(D^T D + I\right) \\ &= \left(D^T D\right)^T + I^T \\ &= D^T D^{TT} + I \\ &= D^T D + I \\ &= X, \end{aligned}$$

as required.

11. One could find the inverse of A by reducing $\begin{pmatrix} A & I_3 \end{pmatrix}$; but in this case A is so simple that its inverse,

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & -1 & 0 \end{pmatrix},$$

can be found by inspection.

12. a. A is the product of the inverses of the elementary matrices corresponding (in order) to the elementary row operations in the given reduction of A to I_2; *i.e.*,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

b. The determinant of A is the product of the determinants of the elementary matrices given in Part a; *i.e.*, -6.

13. One has

$$\begin{pmatrix} 2 & -3 & 1 & 2 \\ 4 & -4 & 5 & 3 \\ -6 & 13 & 4 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 1 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

via the rough work

$$\boxed{\begin{matrix} 2 & 3 & -1 \\ 4 & 7 & 0 \end{matrix}} \rightsquigarrow \boxed{\begin{matrix} 1 & 2 \end{matrix}}.$$

14. a. The equation in question is equivalent to

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y + S \end{pmatrix},$$

(by computing the product on the right hand side) and then comparing blocks gives

$$A_{12} = A_{11}Y, \quad \text{or} \quad Y = A_{11}^{-1}A_{12},$$
$$A_{21} = XA_{11}, \quad \text{or} \quad X = A_{21}A_{11}^{-1},$$

and finally

$$\begin{aligned} A_{22} &= XA_{11}Y + S \\ &= A_{21}A_{11}^{-1}\,A_{11}\,A_{11}^{-1}A_{12} \\ &= A_{21}A_{11}^{-1}A_{12} + S, \end{aligned}$$

and so

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

b. Since

$$L = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$$

are invertible, the rank of A is equal to the rank of

$$D = \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix},$$

which is equal to $\operatorname{rank} A_{11} + \operatorname{rank} S = 4 + \operatorname{rank} S$. Therefore, the rank of S is 3 and, since S is a 5×9 matrix, the nullity of S is $9 - 3 = 6$ by the rank formula. (This last part could also be deduced from the fact that, since A_{11} is invertible, the nullity of A is equal to the nullity of S.)

c. If $A^T = A$ then $A_{11}^T = A_{11}$, $A_{22}^T = A_{22}$ and $A_{12}^T = A_{21}$, and so

$$\begin{aligned} S^T &= (A_{22} - A_{21}A_{11}^{-1}A_{12})^T \\ &= A_{22}^T - A_{12}^T(A_{11}^T)^{-1})A_{21}^T \\ &= A_{22} - A_{21}A_{11}^{-1}A_{12} \\ &= S; \end{aligned}$$

therefore, the statement in question is true.

d. If A is invertible then A_{22} need not be invertible; a simple counterexample is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

in which A is partitioned into four 1×1 matrices. So the statement is false.

e. Since the matrices L and U from Part b are invertible, A is invertible if, and only if, the matrix D of Part b is invertible. Since D and A_{11} are square, S is square. The inverse of D will be such that

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11} \\ SB_{21} & SB_{22} \end{pmatrix},$$

which implies that $B_{11} = A_{11}^{-1}$ and $B_{12} = 0$ since A_{11} is invertible, and $B_{22} = S^{-1}$ by the Invertible Matrix Theorem (and therefore $B_{21} = 0$). So S in invertible, and the statement is true.

15. a. V is the kernel of the linear transformation $\mathrm{M}_{m\times n} \to \mathbb{R}^m$ defined by $A \rightsquigarrow A\mathbf{b}$, which implies that V is a subspace of $\mathrm{M}_{m\times n}$.

b. A 2×2 matrix $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix}$ satisfies $A\mathbf{b} = \mathbf{0}$ if, and only if, $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{0}$, *i.e.*, A has the form $\mathbf{u}\mathbf{v}^T$, where $\mathbf{u} \in \mathbb{R}^2$ and

$$\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

and so

$$\mathscr{B} = \{\mathbf{e}_1\mathbf{v}^T, \mathbf{e}_2\mathbf{v}^T\} = \left\{ \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} \right\}$$

is a basis of V.

c. If $A \in V$, then by Part b, $A^2 = \mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T = (\mathbf{v}^T\mathbf{u})(\mathbf{u}\mathbf{v}^T) = (\mathbf{v}^T\mathbf{u})A$, *i.e.*, A^2 is a scalar multiple of A.

16. a. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ (for which each side of the equation defining S is zero) belong to S, but $\mathbf{e}_1 + \mathbf{e}_2$ (for which the left side of the equation defining S is one and the right side is zero) does not belong to S, it follows that S is not closed under addition.

b. If $\mathbf{x}$ belongs to S and α is any scalar, then $x_1x_2 = x_3^2$, which implies that $(\alpha x_1)(\alpha x_2) = \alpha^2 x_1 x_2 = \alpha^2 x_3^2 = (\alpha x_3)^2$, so $\alpha\mathbf{x}$ belongs to S. Therefore, S is closed under scalar multiplication.

17. a. Reducing A gives

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and therefore the first two columns of A form a basis of the column space of A.

b. From the result of Part a it follows that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_3$ and $\mathbf{a}_4 = 3\mathbf{a}_1 - \mathbf{a}_2$, and so

$$\left\{ \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis of the null space of A.

c. Reducing B gives

$$B \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

from which it follows that $\{\mathbf{r}_1, \mathbf{r}_2\}$ is a basis of the row space of B, where

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

d. By inspection, one finds $\mathbf{a}_1 = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{a}_2 = \mathbf{r}_2$ and $\mathbf{r}_1 = \mathbf{a}_1 + \mathbf{a}_2$, from which it follows that $\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2\} = \operatorname{Span}\{\mathbf{r}_1, \mathbf{r}_2\} = \operatorname{Row} B$.

18. a. If A is an $n \times n$, $\dim \operatorname{Col} A = r$ and $\operatorname{Nul} A = \operatorname{Col} A$, then by the rank formula, $n = \dim \operatorname{Col} A + \dim \operatorname{Nul} A = 2r$, which implies that n is even.

b. If $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$ is an $n \times n$ matrix such that $\operatorname{Col} A = \operatorname{Nul} A$ then, since every column of A belongs to the null space of A, column j of A^2 is $A\mathbf{a}_j = \mathbf{0}$, for $1 \leqslant j \leqslant n$, and therefore $A^2 = 0$.

19. a. The area of the shaded face is $\|\mathbf{n}\| = \sqrt{147} = 7\sqrt{3}$, where

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -11 \\ -5 \\ 1 \end{pmatrix}.$$

b. The origin lies on the plane ϖ containing the shaded face, and $-\mathbf{n}$ (where $\mathbf{n}$ is from Part a) is normal to ϖ, so $(-\mathbf{n})^T\mathbf{x} = 0$, or $11x_1 + 5x_2 - x_3 = 0$, is an equation of the plane ϖ.

c. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a right-handed basis of $\mathbb{R}^3$, the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is equal to $\det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} = (\mathbf{u} \times \mathbf{v})^T\mathbf{w} = 40$.

d. From the figure it is clear that

$$\mathbf{a} = \mathbf{u} + \mathbf{v} + \mathbf{w} = \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix}.$$

e. The line in question contains $\mathbf{a}$ and is parallel to $\mathbf{u}$, so it consists of all vectors of the form $\mathbf{a} + t\mathbf{u}$, where t is a real number; probably $\mathbf{x} = \mathbf{a} + t\mathbf{u}$ is the requested equation.

20. a. The distance between ℓ_1 and ℓ_2 is equal to the length of the orthogonal projection of $\mathbf{p}$ onto $\mathbf{n} = \mathbf{e}_2 \times \mathbf{u}$. The cross product

$$\mathbf{n} = \mathbf{e}_2 \times \mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

the projection

$$\operatorname{proj}_{\mathbf{n}} \mathbf{p} = \frac{\mathbf{n}^T\mathbf{p}}{\mathbf{n}^T\mathbf{n}}\mathbf{n} = \tfrac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

and so the distance between the lines is $\|\operatorname{proj}_{\mathbf{n}} \mathbf{p}\| = \frac{1}{2}\sqrt{2}$.

b. The point on ℓ_2 which is closest to the origin is the orthogonal component of the projection of $\mathbf{p}$ onto $\mathbf{u}$, *i.e.*,

$$\operatorname{perp}_{\mathbf{u}} \mathbf{p} = \mathbf{p} - \frac{\mathbf{u}^T\mathbf{p}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - \tfrac{4}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \tfrac{1}{3}\begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}.$$

21. a. The two unit vectors orthogonal to $\mathbf{u}$ are

$$\frac{\pm 1}{\sqrt{1+a^2}}\begin{pmatrix} a \\ -1 \end{pmatrix}.$$

b. The length of $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ is

$$\frac{|\mathbf{u}^T\mathbf{v}|}{\|\mathbf{v}\|} = \frac{2|a|}{\sqrt{1+a^2}}.$$

c. The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u}^T\mathbf{v}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \frac{2a}{1+a^2}\begin{pmatrix} a \\ 1 \end{pmatrix},$$

and so

$$\lim_{a\to\infty} \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$