1. Given the homogeneous system with matrix equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 0 & 2 & -1 & 0 \\ 1 & 1 & -5 & 5 & 1 \\ 2 & 2 & -10 & 10 & 3 \\ 2 & 1 & -7 & 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

a. Write the solution of the system in parametric vector form.
b. Write the zero vector as a non-trivial linear combination of the columns of A.

2. Find a quadratic polynomial p such that $p(2) = 0$, $p(-2) = 32$ and $p'(1) = -7$.

3. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} k \\ 0 \\ 2k+3 \end{pmatrix}$.

For which value(s) of k is $\operatorname{Span} S$: a. $\mathbb{R}^3$? b. a plane in $\mathbb{R}^3$? c. a line in $\mathbb{R}^3$?

4. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + 2y \\ 2x - 3y \end{pmatrix}$.
a. Find the standard matrix of T_1.
b. Find all values of k for which $T_1(\ell)$ a horizontal line, where ℓ has parametric representation

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ k \end{pmatrix}.$$

c. If $T_1 \circ T_2$ is a linear transformation with standard matrix $\begin{pmatrix} 1 & -2 & -3 \\ -3 & 5 & 7 \end{pmatrix}$, then

i. identify the domain and codomain of T_2, and ii. find the standard matrix of T_2.

5. Suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, and that $\mathbf{x} = 2\mathbf{u} + 3\mathbf{w}$ and $\mathbf{y} = \mathbf{v} + 2\mathbf{w}$. Prove that $\{\mathbf{u}, \mathbf{x}, \mathbf{y}\}$ is linearly independent.

6. Find an LU factorization of $\begin{pmatrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{pmatrix}$.

7. Let $A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix}$ and $B = \begin{pmatrix} a + 2b + 4c & d + 2e + 4f & g + 2h + 4k \\ 3a + 4b + 7c & 3d + 4e + 7f & 3g + 4h + 7k \\ 5a + 7b + 8c & 5d + 7e + 8f & 5g + 7h + 8k \end{pmatrix}$.

a. Find a matrix C such that $B = CA$.
b. Find a scalar λ such that $\det B = \lambda \det A$ for all possible choices of A.

8. Let A be a 3×3 matrix with $\det A = -2$.
a. Find $\det\big(A^T A^2 (-2A)^{-1}\big)$. b. Find $\det\big(\operatorname{adj}(2A)\big)$.

9. a. Given matrices A, B and C with A and B invertible, find matrices W, X, Y and Z such that

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix} \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

b. Use the above result to find D^{-1}, where

$$D = \begin{pmatrix} 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 & 0 \\ 1 & 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

10. Use Cramer's Rule to solve the linear system: $\begin{aligned} 7x - 9y &= 11 \\ 4x + 5y &= -2. \end{aligned}$

11. Simplify the matrix expression $\big(B(B+I)^{-1}\big)^{-1} - B^{-1}$.

12. You are given a the matrix A and its reduced echelon form R:

$$A = \begin{pmatrix} 3 & 6 & 2 & 1 & 5 & 2 & 2 \\ 1 & 2 & 1 & 0 & 2 & 0 & -3 \\ 1 & 2 & 0 & 1 & 1 & 1 & 4 \\ 1 & 2 & 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 3 & 0 & -1 \end{pmatrix}; \quad R = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a. For which value of n is $\operatorname{Row} A$ a subspace of $\mathbb{R}^n$?
b. Without calculation, give a basis of $\operatorname{Row} A$.
c. For which value of m is $\operatorname{Col} A$ a subspace of $\mathbb{R}^m$?
d. Without calculation, give a basis of $\operatorname{Col} A$.
e. What is the rank of A?
f. What is the dimension of $\operatorname{Nul} A^T$?

13. Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0 \right\}.$$

a. Is $\mathbf{0} \in W$? Justify your answer.
b. Is W closed under scalar multiplication? Justify your answer.
c. Is W closed under addition? Justify your answer.
d. Is W a subspace of $\mathbb{R}^2$? Explain.

14. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find a 2×2 matrix B such that $AB = 0$ but $BA \neq 0$.

15. Suppose that A and B are $n \times n$ matrices such that $AB = 0$ and $BA \neq 0$.
a. Show that each column of B is in the null space of A.
b. What that even though $BA \neq 0$, it must be true that $(BA)^2 = 0$.
c. Show that B is not invertible.

16. Let V be the set of all 2×2 matrices X such that $AX = 0$, where

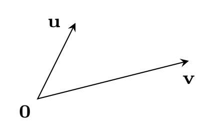
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Explain why V is a subspace of $M_{2\times 2}$, find a basis of V.

17. Let $V = \{\, p \in \mathbb{P}_2 : p(2) = 0 \,\}$. Explain why V is a subspace of $\mathbb{P}_2$, and find a basis of V and the dimension of V.

18. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
a. What is the dimension of the range of T if T is injective? Explain.
b. What is the dimension of the kernel of T if T is surjective? Explain.
c. Prove that if the rank of T is m then there is a linear transformation $S : \mathbb{R}^m \to \mathbb{R}^n$ such that $T \circ S = \mathrm{id}_{\mathbb{R}^m}$.

19. Given this picture



of two vectors in $\mathbb{R}^n$, draw a picture of each of the following.

a. $\{\, (1-t)\mathbf{u} + t\mathbf{v} : 0 \leqslant t \leqslant 1 \,\}$ b. $\{\, s\mathbf{u} + t\mathbf{v} : 0 \leqslant s \leqslant 1,\ 0 \leqslant t \leqslant \frac{1}{2} \,\}$
c. $\operatorname{proj}_{\mathbf{v}} \mathbf{u} + \operatorname{proj}_{\mathbf{u}} \mathbf{v}$ d. $\operatorname{proj}_{\mathbf{v}} \mathbf{u} - \operatorname{perp}_{\mathbf{v}} \mathbf{u}$

20. Let

$$\ell = \left\{ \begin{pmatrix} -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

a. Plot ℓ (*i.e.*, draw a rough picture of ℓ).
b. Find the distance between ℓ and the origin.
c. For which values of a and b, if any, is the line with parametric representation

$$\begin{pmatrix} 1 \\ a \end{pmatrix} + t \begin{pmatrix} 1 \\ b \end{pmatrix}$$

equal to ℓ?
d. Where does ℓ intersect the x_1-axis?
e. What is the cosine of the angle between ℓ and the x_1-axis.

21. Let $V = \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

a. Find an implicit equation defining V.
b. Find a parametric representation of the line orthogonal to V which contains the origin.
c. For which values of α, if any, does $\mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k}$ belong to V?

22. Give two different unit vectors which are orthogonal to both

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

23. What are the possible values of the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ if

$$\frac{1}{\mathbf{u} \cdot \mathbf{v}} (\mathbf{u} \times \mathbf{v})$$

is a unit vector?

1. Below is the coefficient matrix A of the system, and its reduced echelon form.

$$\begin{pmatrix} -1 & 0 & 2 & -1 & 0 \\ 1 & 1 & -5 & 5 & 1 \\ 2 & 2 & -10 & 10 & 3 \\ 2 & 1 & -7 & 6 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

a. The first, second and fifth columns of A are pivot columns, and the non-pivot columns of A are given by

$$\mathbf{a}_3 = -2\mathbf{a}_1 - 3\mathbf{a}_2 \qquad \text{and} \qquad \mathbf{a}_4 = \mathbf{a}_1 + 4\mathbf{a}_2,$$

so the general solution of $A\mathbf{x} = \mathbf{0}$ is the set of all vectors of the form

$$s\begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where s and t are real numbers.

b. Either of the relations displayed in Part a (or any combination thereof) expresses the zero vector as a non-trivial linear combination of the columns of A; *e.g.*, $2\mathbf{a}_1 + 3\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$.

2. The polynomial $p(x) = a_0 + a_1x + a_2x^2$ satisfies $p(2) = 0$, $p(-2) = 32$ and $p'(1) = -7$ if, and only if,

$$\begin{aligned} a_0 + 2a_1 + 4a_2 &= 0, \\ a_0 - 2a_1 + 4a_2 &= 32 \text{ and} \\ a_1 + 2a_2 &= -7, \end{aligned}$$

where the last equation uses $p'(x) = a_1 + 2a_2x$. Reducing the coefficient matrix of this linear system gives

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & -2 & 4 & 32 \\ 0 & 1 & 2 & -7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right),$$

and therefore, $p(x) = 14 - 8x + \frac{1}{2}x^2$.

3. Row reduction gives

$$A = \begin{pmatrix} \mathbf{v}_2 & \mathbf{v}_1 & \mathbf{v}_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & -k^2 + 2k + 3 \end{pmatrix},$$

and therefore: a. $\operatorname{Span} S = \mathbb{R}^3$ if, and only if, A has three pivot columns, *i.e.*, $0 \neq -k^2 + 2k + 3 = (3 - k)(1 + k)$, or $k \neq -1, 3$; b. $\operatorname{Span} S$ is a plane in $\mathbb{R}^3$ if, and only if, A has two pivot columns, *i.e.*, $0 = (3 - k)(1 + k)$, or $k = -1, 3$; c. there are no values of k such that $\operatorname{Span} S$ is a line in $\mathbb{R}^3$ since in any case A has at least two pivot columns.

4. a. The standard matrix of T_1 is $A = \begin{pmatrix} T_1(\mathbf{e}_1) & T_1(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$.

b. $T_1(\ell)$ is a horizontal line if, and only if, there is a non-zero scalar α such that

$$T_1\begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} -1 + 2k \\ 2 - 3k \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix};$$

i.e., if, and only if, $k = \frac{2}{3}$.

c. i. The domain of T_2 is the domain of $T_1 \circ T_2$, which is $\mathbb{R}^3$. The codomain of T_2 is the domain of T_1, which is $\mathbb{R}^2$. ii. Since $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible (for example, because $\det A \neq 0$) and $T_1 \circ T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ is linear, it follows that $T_2 = T_1^{-1} \circ (T_1 \circ T_2) : \mathbb{R}^3 \to \mathbb{R}^2$ is linear, and its standard matrix is

$$A^{-1}\left[T_1 \circ T_2\right] = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -3 \\ -3 & 5 & 7 \end{pmatrix} = \begin{pmatrix} -3 & 4 & 5 \\ -1 & 1 & 1 \end{pmatrix}.$$

5. If α, β and γ are scalars and $\alpha\mathbf{u} + \beta\mathbf{x} + \gamma\mathbf{y} = \mathbf{0}$, then the expressions defining $\mathbf{x}$ and $\mathbf{y}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ give $\alpha\mathbf{u} + \beta(2\mathbf{u} + 3\mathbf{w}) + \gamma(\mathbf{v} + 2\mathbf{w}) = \mathbf{0}$, or $(\alpha + 2\beta)\mathbf{u} + \gamma\mathbf{v} + (3\beta + 2\gamma)\mathbf{w} = \mathbf{0}$, which implies that $\alpha + 2\beta = 0$, $\gamma = 0$ and $3\beta + 2\gamma = 0$ because $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. Since $\gamma = 0$ and $3\beta + 2\gamma = 0$ it follows that $\beta = 0$, and therefore, since $\alpha + 2\beta = 0$, that $\alpha = 0$. This shows that $\alpha\mathbf{u} + \beta\mathbf{x} + \gamma\mathbf{y} = \mathbf{0}$ implies that $\alpha = 0$, $\beta = 0$ and $\gamma = 0$; *i.e.*, that $\{\mathbf{u}, \mathbf{x}, \mathbf{y}\}$ is linearly independent, as required.

6. One has

$$\begin{pmatrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

via the rough work

$$\begin{matrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} -5 \\ -10 \\ -15. \end{matrix}$$

7. Since $B = CA$, where

$$C = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 5 & 7 & 8 \end{pmatrix}$$

it follows that $\det B = (\det C)(\det A)$, so that

$$\lambda = \det C = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 5 & 7 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2 & -5 \\ 5 & -3 & -12 \end{vmatrix} = 3\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 9$$

(using column reduction, multilinearity and then cofactor expansion).

8. Multilinearity, product preservation and invariance under transposition of the determinant gives

a. $\det\bigl(A^TA^2(-2A)^{-1}\bigr) = (\det A)(\det A)^2(-2)^{-3}(\det A)^{-1} = -\frac{1}{2}$, and

b. $\det\bigl(\operatorname{adj}(2A)\bigr) = \det\bigl(\det(2A)(2A)^{-1}\bigr) = (2^3)^3(\det A)^3 2^{-3}(\det A)^{-1} = 256$.

9. a. Expanding the product on the left side of the given equation yields

$$\begin{pmatrix} AY & AZ \\ BW + CY & BX + CZ \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Since A is invertible, comparing the upper blocks gives $Y = A^{-1}$ and $Z = 0$. Next, comparing lower left blocks gives $BW + CY = 0$, *i.e.*, $BW = -CY$, or $W = -B^{-1}CY$ since B is invertible. Finally, comparing lower right blocks gives $I = BX + CZ = BX$ (since $Z = 0$), and so $X = B^{-1}$. Therefore,

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}^{-1} = \begin{pmatrix} -B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & 0 \end{pmatrix}.$$

b. Partition the given matrix D as in Part a so that

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

The formula for the inverse of a 2×2 matrix, and row reducing $\begin{pmatrix} B & I_3 \end{pmatrix}$, gives

$$A^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} \qquad \text{and} \qquad B^{-1} = \begin{pmatrix} 0 & 1 & -3 \\ 1 & -2 & 6 \\ 0 & 0 & 1 \end{pmatrix};$$

and so

$$-B^{-1}CA^{-1} = -\tfrac{1}{2}\begin{pmatrix} 2 & -4 \\ 0 & 0 \\ -1 & 3 \end{pmatrix}$$

by a direct calculation. Therefore,

$$D^{-1} = \begin{pmatrix} -1 & 2 & 0 & 1 & -3 \\ 0 & 0 & 1 & -2 & 6 \\ \frac{1}{2} & -\frac{3}{2} & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 & 0 \end{pmatrix}$$

by the result from Part a.

10. Cramer's Rule gives

$$x = \frac{\begin{vmatrix} 11 & -9 \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 7 & -9 \\ 4 & 5 \end{vmatrix}} = \frac{37}{71} \qquad \text{and} \qquad y = \frac{\begin{vmatrix} 7 & 11 \\ 4 & -2 \end{vmatrix}}{\begin{vmatrix} 7 & -9 \\ 4 & 5 \end{vmatrix}} = -\frac{58}{71}.$$

11. Since the inverse of a product of two matrices is the product of their inverses, taken in the opposite order, and the inverse of the inverse of a matrix is the original matrix, one has

$$\big(B(B+I)^{-1}\big)^{-1} = \big((B+I)^{-1}\big)^{-1}B^{-1} = (B+I)B^{-1}.$$

Therefore, the expression in question is equal to

$$(B+I)B^{-1} - B^{-1} = (B+I-I)B^{-1} = BB^{-1} = I,$$

by distributivity and the definition of the inverse of a matrix.

12. a. Row A is a subspace of $\mathbb{R}^7$.

b. A basis of Row A is the list of transposes of the non-zero rows of the reduced echelon form (or any echelon form) of A:

$$\left\{ \begin{pmatrix}1\\2\\0\\0\\3\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\-1\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\1\\4\end{pmatrix} \right\}.$$

c. Col A is a subspace of $\mathbb{R}^5$.

d. A basis of Col A is the list of pivot columns of A:

$$\left\{ \begin{pmatrix}3\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\1\\0\end{pmatrix} \right\}.$$

e. The rank of A is four.

f. $\dim \operatorname{Nul} A^T = 5 - \operatorname{rank} A = 5 - 4 = 1$.

13. a. The zero vector belongs to W.

b. If $\mathbf{x} \in W$ and α is any scalar, then whichever entry of $\mathbf{x}$ is zero remains so when multiplied by α, so $\alpha\mathbf{x} \in W$. Thus, W is closed under scalar multiplication.

c. Since $\mathbf{e}_1 \in W$ and $\mathbf{e}_2 \in W$ but $\mathbf{e}_1 + \mathbf{e}_2 \notin W$, W is not closed under addition.

d. By Part c, W is not closed under addition, so W is not a subspace of $\mathbb{R}^2$.

14. If $B = \begin{pmatrix}\mathbf{e}_1 - \mathbf{e}_2 & \mathbf{0}\end{pmatrix}$, then

$$AB = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

but

$$BA = \begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\-1 & -1\end{pmatrix} \neq \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix},$$

as required.

15. a. If $\mathbf{b}_j$ denotes column j of B then column j of $AB = 0$ is $A\mathbf{b}_j = \mathbf{0}$, which implies that $\mathbf{b}_j \in \operatorname{Nul} A$.

b. One has (with *all* of the details)

$$(BA)^2 = (BA)(BA) = \big((BA)B\big)A = \big(B(AB)\big)A = (B0)A = 0A = 0,$$

as required.

c. If B is invertible then $0 \neq BA = BABB^{-1} = B0B^{-1} = 0$, which is impossible.

16. V is a subspace of $M_{2\times 2}$ because V is the kernel of the linear transformation $T\colon M_{2\times 2} \to M_{2\times 2}$ defined by $T(X) = AX$. Next, if $AX = 0$ then each column of X belongs to $\operatorname{Nul} A = \operatorname{Span}\{\mathbf{e}_1 - \mathbf{e}_2\}$; therefore V is the span of

$$\mathscr{B} = \left\{ \begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & -1\end{pmatrix} \right\}.$$

Since $\mathscr{B}$ is clearly linearly independent, it forms a basis of V.

17. V is a subspace of $\mathbb{P}_2$ because it is the kernel of the linear transformation $T\colon \mathbb{P}_2 \to \mathbb{R}$ defined by $T(p) = p(2)$. Next, since $\{\, p \in \mathbb{P}_2 : p(0) = 0 \,\} = \{\, \alpha t + \beta t^2 : \alpha, \beta \in \mathbb{R} \,\}$ has basis $\{t, t^2\}$, and since replacing t by $t-2$ defines an isomorphism $\mathbb{P}_2 \to \mathbb{P}_2$, it follows that $\{t-2, (t-2)^2\}$ is a basis of V, and the dimension of V is two.

18. a. If T is injective then the dimension of the kernel of T is zero. Since $\dim\ker T + \operatorname{rank} T = n$ by the rank formula, it follows that the dimension of the range of T (*i.e.*, the rank of T) is n.

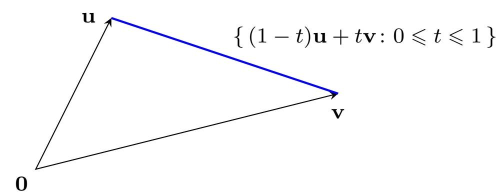
b. If T is surjective then its rank is m, so (as in Part a) the rank formula implies that the dimension of the kernel of T is $n - m$.

c. If the rank of T is m then T is surjective, and so there are $\mathbf{x}_i \in \mathbb{R}^n$ such that $T(\mathbf{x}_i) = \mathbf{e}_i$ for $i = 1, \dots, m$. Let $S\colon \mathbb{R}^m \to \mathbb{R}^n$ be the unique linear transformation such that $S(\mathbf{e}_i) = \mathbf{x}_i$ for $i = 1, \dots, m$ (*i.e.*, column i of the standard matrix of S is $\mathbf{x}_i$). If $\mathbf{y} = y_1\mathbf{e}_1 + \cdots + y_m\mathbf{e}_m$ is any vector in $\mathbb{R}^m$, then

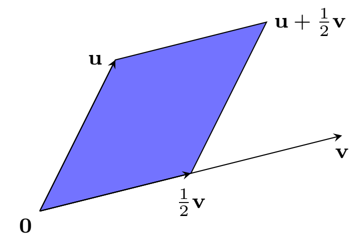
$$\begin{aligned} T\circ S(\mathbf{y}) &= y_1 T\circ S(\mathbf{e}_1) + \cdots + y_m T\circ S(\mathbf{e}_m) \\ &= y_1 T(\mathbf{x}_1) + \cdots + y_m T(\mathbf{x}_m) \\ &= y_1\mathbf{e}_1 + \cdots + y_m\mathbf{e}_m \\ &= \mathbf{y}, \end{aligned}$$

by the linearity of $T\circ S$ (since S and T are linear, so is $T\circ S$) and the definition of S. This shows that $T\circ S = \mathrm{id}_{\mathbb{R}^m}$, as required.

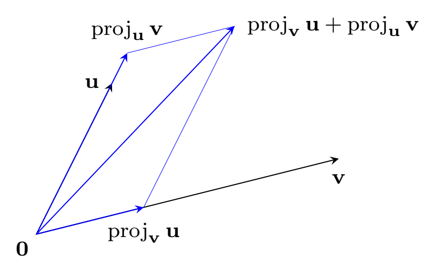
19. a. $\{\, (1-t)\mathbf{u} + t\mathbf{v} : 0 \leqslant t \leqslant 1 \,\}$ is the segment joining $\mathbf{u}$ and $\mathbf{v}$ (parametrized in the direction going from $\mathbf{u}$ to $\mathbf{v}$).



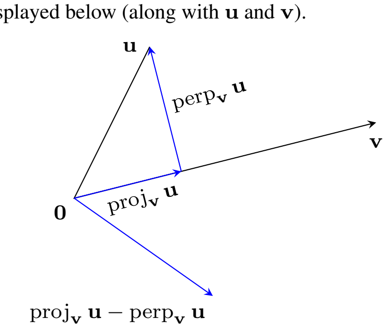
b. $\{\, s\mathbf{u} + t\mathbf{v} : 0 \leqslant s \leqslant 1,\ 0 \leqslant t \leqslant \frac{1}{2} \,\}$ is the (interior and boundary of the) parallelogram formed by the segments joining the origin to $\mathbf{u}$ and to $\frac{1}{2}\mathbf{v}$.



c. The projections and their sum are displayed below (together with $\mathbf{u}$ and $\mathbf{v}$).



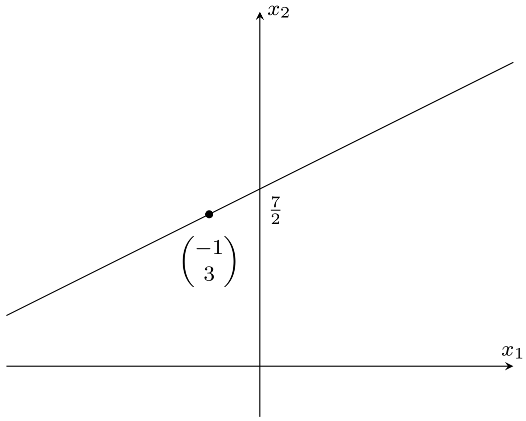
d. The projections (the orthogonal component translated by $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$) and their difference are displayed below (along with $\mathbf{u}$ and $\mathbf{v}$).



20. a. Below is a rough sketch of $\ell = \{\, \mathbf{p} + t\mathbf{v} \colon t \in \mathbb{R} \,\}$, where

$$\mathbf{p} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix};$$

the x_1 intercept (not displayed) is at -7.



b. The distance from $\mathbf{p}$ to the origin is equal to the area of parallelogram formed by $\mathbf{p}$ and $\mathbf{v}$, divided by the length of $\mathbf{v}$, which is given by

$$\frac{|\det\begin{pmatrix}\mathbf{p} & \mathbf{v}\end{pmatrix}|}{\|\mathbf{v}\|} = \tfrac{7}{5}\sqrt{5}.$$

(*Note*: The distance is also given by the length of $\operatorname{perp}_{\mathbf{v}} \mathbf{p}$.)

c. For the given line to be equal to ℓ, its direction vector must be parallel to $\mathbf{v}$, which gives $b = \frac{1}{2}$, and the equation

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$$

must be consistent, which gives $a = 4$.

d. ℓ intersects the x_1-axis at the point $(-7, 0)$.

e. The cosine of the angle between ℓ and the x_1-axis is

$$\frac{\mathbf{v}^T\mathbf{e}_1}{\|\mathbf{v}\|\|\mathbf{e}_1\|} = \tfrac{2}{5}\sqrt{5}.$$

21. a. A normal vector to V is given by

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix},$$

and an implicit equation defining V is $\mathbf{n}^T\mathbf{x} = 0$, or $x_1 - 2x_2 - 3x_3 = 0$.

b. The line which is orthogonal to V and contains the origin is the set of all vectors of the form $t\mathbf{n}$, where t is a real number and $\mathbf{n}$ is the normal vector found in Part a.

c. $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k}$ belongs to V if, and only if, $\mathbf{n}^T\mathbf{w} = 0$, or $\alpha = -1$.

22. The unit vectors $\pm\hat{\mathbf{u}}$ are orthogonal to the given vectors, where

$$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ -7 \end{pmatrix}, \quad \text{and so} \quad \hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \tfrac{1}{30}\sqrt{6}\begin{pmatrix} -1 \\ 10 \\ -7 \end{pmatrix}.$$

23. First of all, if

$$\mathbf{w} = \frac{1}{\mathbf{u}\cdot\mathbf{v}}(\mathbf{u} \times \mathbf{v})$$

is a unit vector then neither $\mathbf{u}$ nor $\mathbf{v}$ is $\mathbf{0}$ (otherwise, $\mathbf{u}\cdot\mathbf{v}$ would be zero and so $\mathbf{w}$ would be undefined). Next, observe that if $\|\mathbf{w}\| = 1$ then $\mathbf{u}\cdot\mathbf{v} = \pm\|\mathbf{u}\times\mathbf{v}\|$, *i.e.*, $\|\mathbf{u}\|\|\mathbf{v}\|\cos\vartheta = \pm\|\mathbf{u}\|\|\mathbf{v}\|\sin\vartheta$, where ϑ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Since $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors, it follows that $\|\mathbf{u}\|\|\mathbf{v}\| \neq 0$, and so $\cos\vartheta = \pm\sin\vartheta$. Therefore, $\vartheta = \frac{1}{4}\pi$ or $\vartheta = \frac{3}{4}\pi$ (because $0 \leqslant \vartheta \leqslant \pi$).