1. Use linear algebra to balance the chemical equation

$$\mathrm{NH_3} + \mathrm{Br_2} \longrightarrow \mathrm{NH_4Br} + \mathrm{N_2}.$$

2. In this problem you are given the matrix $A = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6\end{pmatrix}$, with the fourth column unspecified, and its reduced echelon form R:

$$A = \begin{pmatrix} 1 & -7 & 4 & a & 5 & 2 \\ -1 & 7 & 2 & b & 2 & -1 \\ 2 & -14 & 3 & c & 3 & 2 \\ 3 & -21 & -2 & d & 1 & 6 \end{pmatrix};$$

$$R = \begin{pmatrix} 1 & -7 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a. Solve the equation $A\mathbf{x} = \mathbf{0}$.

b. What is $\mathbf{a}_4$? (That is, find a, b, c and d.)

c. Given that $\mathbf{b} = \mathbf{a}_2 - 2\mathbf{a}_4 + 3\mathbf{a}_6$, solve the equation $A\mathbf{x} = \mathbf{b}$.

d. What is the dimension of $\mathrm{Nul}(A^T)$?

3. Let

$$\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

a. Find a matrix $A_\mathbf{v}$ so that $\mathbf{v} \times \mathbf{x} = A_\mathbf{v}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

b. Use the identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}^T\mathbf{w})\mathbf{v} - (\mathbf{u}^T\mathbf{v})\mathbf{w}$ to show that

$$A_\mathbf{v}^2\mathbf{x} = (\mathbf{v}^T\mathbf{x})\mathbf{v} - \|\mathbf{v}\|^2\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^3$.

c. If $\mathbf{n}$ is a unit vector, show that the orthogonal projection of $\mathbf{x}$ onto the subspace of $\mathbb{R}^3$ orthogonal to $\mathbf{n}$ is given by $-A_\mathbf{n}^2\mathbf{x}$.

4. Let

$$A = \begin{pmatrix} 2 & 1 \\ c & d \end{pmatrix}.$$

a. Find c and d so that $A^2 = 0$.

b. Find c and d so that $A^2 = I$.

c. Find all values of c and d so that A^2 is symmetric.

5. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 15 & 0 & 1 \end{pmatrix}.$$

a. Find the inverse of A.

b. Write A as a product of elementary matrices.

6. Let A and B be invertible $n \times n$ matrices, and C and D be singular $n \times n$ matrices. Fill in the blanks. The missing word is **must**, **might** or **cannot**.

a. $A + C$ __________ be invertible.

b. A^TA __________ be invertible.

c. AC and BC __________ have the same determinant.

d. $\mathrm{Col}\, C$ __________ be equal to $\mathrm{Col}\, D$.

e. $\mathrm{Nul}\, A$ __________ be equal to $\mathrm{Nul}\, B$.

f. The columns of D __________ be linearly independent.

7. Find an LU factorization of

$$\begin{pmatrix} 3 & -1 & 3 \\ 15 & -3 & 13 \\ 12 & 2 & 10 \end{pmatrix}.$$

8. Let A be a 5×5 matrix with $\det A = 2$ and let I be the 5×5 identity matrix. Furthermore, assume $A = LU$ where L is unit lower triangular and U is upper triangular. Calculate: a. $\det U$; b. $\det(3A^{-1}A^T)$; c. $\det(L + I)$.

9. Verify that Cramer's Rule applies to the following system and then use it to solve for x_2 only.

$$\begin{aligned} 3x_1 \quad - x_3 \quad &= 2 \\ 2x_2 \quad + 5x_4 &= 0 \\ -4x_1 \quad + 2x_3 \quad &= -1 \\ -5x_2 \quad - 5x_4 &= 4 \end{aligned}$$

10. a. Is $\mathscr{S}_1 = \{\, p \in \mathbb{P}_2 \colon p(0) \geqslant 0 \,\}$ a subspace of $\mathbb{P}_2$? If so, find a basis of $\mathscr{S}_1$. If not, explain why not.

b. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let $\mathscr{S}_2 = \{\, X \in M_{2\times 2} \colon XA = AX \,\}$. Is $\mathscr{S}_2$ a subspace of $M_{2\times 2}$? If so, find a basis of $\mathscr{S}_2$. If not, explain why not.

11. Let $T \colon \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(p) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}.$$

a. If $p(t) = t^2 - 1$, find $T(p)$.

b. Find a basis of the kernel of T.

12. Suppose that A is a 5×5 matrix of rank 3, and let

$$B = \begin{pmatrix} A & A \\ A & A \end{pmatrix}.$$

a. What is the rank of B?

b. What is the dimension of the null space of B.

13. Let ℓ_1 be the line given by $\mathbf{p} + t\mathbf{u}$ and let ℓ_2 be the line given by $\mathbf{q} + t\mathbf{v}$, where

$$\mathbf{p} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

a. Find the point of intersection of ℓ_1 and ℓ_2.

b. Find the cosine of the angle between ℓ_1 and ℓ_2.

c. Find a normal equation of the plane containing ℓ_1 and ℓ_2.

14. Let ϖ be the plane in $\mathbb{R}^3$ defined by $x - 2y + z = 8$.

a. Find the distance from the origin to ϖ.

b. Find an equation of the line through the origin perpendicular to ϖ.

c. Find the point on ϖ which is closest to the origin.

15. Suppose that $\mathrm{proj}_\mathbf{w}(\mathbf{u} + \mathbf{v}) = \mathrm{proj}_\mathbf{w}(\mathbf{u})$, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\mathbf{w} \neq \mathbf{0}$. Show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

16. a. Let

$$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Find two vectors $\mathbf{w}$ on the x_1-axis such that the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is 10.

b. Suppose that the volume of the parallelepiped formed by three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 6. What is the volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}+\mathbf{b}+\mathbf{c}$?

17. a. Is the following statement true or false? Justify your answer.

If $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are linearly dependent, then either $\{\mathbf{a}, \mathbf{u}\}$ or $\{\mathbf{a}, \mathbf{v}\}$ must be linearly dependent.

b. Suppose that $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are linearly independent. Show that either $\{\mathbf{a}, \mathbf{u}\}$ or $\{\mathbf{a}, \mathbf{v}\}$ must be linearly independent.

18. Let

$$A = \begin{pmatrix} I & \mathbf{u} \\ \mathbf{u}^T & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix and $\mathbf{u}$ is a unit vector in $\mathbb{R}^n$.

a. Find A^2. b. Find A^{-1}. c. Find $\det A$.

1. If the entries in a vector denote, respectively, the number of Nitrogen, Hydrogen and Bromine atoms in a given species, then the chemical equation is balanced by positive integer solutions of the vector equation $x_1\mathbf{s}_1 + x_2\mathbf{s}_2 = x_3\mathbf{s}_3 + x_4\mathbf{s}_4$, where

$$\mathbf{s}_1 = \begin{pmatrix}1\\3\\0\end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix}0\\0\\2\end{pmatrix}, \quad \mathbf{s}_3 = \begin{pmatrix}1\\4\\1\end{pmatrix} \quad \text{and} \quad \mathbf{s}_4 = \begin{pmatrix}2\\0\\0\end{pmatrix}.$$

Reducing the matrix $S = \begin{pmatrix}\mathbf{s}_1 & \mathbf{s}_2 & -\mathbf{s}_3 & -\mathbf{s}_4\end{pmatrix}$,

$$\begin{pmatrix}1 & 0 & -1 & -2\\3 & 0 & -4 & 0\\0 & 2 & -1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -8\\0 & 1 & 0 & -3\\0 & 0 & 1 & -6\end{pmatrix},$$

reveals that the general solution of $S\mathbf{x} = \mathbf{0}$ is the set of all vectors of the form $t\mathbf{u}$, where t is a real number and

$$\mathbf{u} = \begin{pmatrix}8\\3\\6\\1\end{pmatrix}.$$

Taking $t = 1$ gives

$$8NH_3 + 3Br_2 \longrightarrow 6NH_4Br + N_2.$$

2. a. The reduced echelon form of A reveals that $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ are pivot columns of A, and that

$$\mathbf{a}_2 = 7\mathbf{a}_1, \quad \mathbf{a}_4 = -3\mathbf{a}_1 + 2\mathbf{a}_3 \quad \text{and} \quad \mathbf{a}_6 = \mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_5,$$

or

$$-7\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}, \quad 3\mathbf{a}_1 - 2\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0} \quad \text{and} \quad -\mathbf{a}_1 + \mathbf{a}_3 - \mathbf{a}_5 + \mathbf{a}_6 = \mathbf{0}.$$

Therefore, the general solution of $A\mathbf{x} = \mathbf{0}$ is the set of all vectors of the form $r\mathbf{u} + s\mathbf{v} + t\mathbf{w}$, where r, s and t are real numbers,

$$\mathbf{u} = \begin{pmatrix}7\\1\\0\\0\\0\\0\end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix}3\\0\\-2\\1\\0\\0\end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix}-1\\0\\1\\0\\-1\\1\end{pmatrix}.$$

b. As in Part a, one has

$$\mathbf{a}_4 = -3\mathbf{a}_2 + 2\mathbf{a}_3 = -3\begin{pmatrix}1\\-1\\2\\3\end{pmatrix} + 2\begin{pmatrix}4\\2\\3\\-2\end{pmatrix} = \begin{pmatrix}5\\7\\0\\-13\end{pmatrix};$$

i.e., $a = 5$, $b = 7$, $c = 0$ and $d = -13$.

c. If

$$\mathbf{p} = \begin{pmatrix}0\\1\\0\\-2\\0\\3\end{pmatrix},$$

then, since $A\mathbf{p} = \mathbf{b}$, the general solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{p} + \operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are from Part a.

d. Since $\operatorname{rank}(A^T) = \operatorname{rank} A = 3$, and A^T has four columns, the rank formula implies that the dimension of $\operatorname{Nul}(A^T)$ is 1.

3. a. The standard matrix of the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{x} \rightsquigarrow \mathbf{v} \times \mathbf{x}$ is

$$A_{\mathbf{v}} = \begin{pmatrix}\mathbf{v}\times\mathbf{e}_1 & \mathbf{v}\times\mathbf{e}_2 & \mathbf{v}\times\mathbf{e}_3\end{pmatrix} = \begin{pmatrix}0 & -\gamma & \beta\\ \gamma & 0 & -\alpha\\ -\beta & \alpha & 0\end{pmatrix}.$$

b. If $\mathbf{x} \in \mathbb{R}^3$ then

$$\begin{aligned} A_{\mathbf{v}}^2\mathbf{x} &= A_{\mathbf{v}}(\mathbf{v}\times\mathbf{x}) = \mathbf{v}\times(\mathbf{v}\times\mathbf{x}) \\ &= (\mathbf{v}^T\mathbf{x})\mathbf{v} - (\mathbf{v}^T\mathbf{v})\mathbf{x} \\ &= (\mathbf{v}^T\mathbf{x})\mathbf{v} - \|\mathbf{v}\|^2\mathbf{x}, \end{aligned}$$

as required.

c. The orthogonal projection of $\mathbf{x}$ onto the subspace of $\mathbb{R}^3$ orthogonal to $\mathbf{n}$ is

$$\begin{aligned} \operatorname{perp}_{\mathbf{n}}(\mathbf{x}) &= \mathbf{x} - \operatorname{proj}_{\mathbf{n}}(\mathbf{x}) \\ &= \mathbf{x} - \frac{\mathbf{n}^T\mathbf{x}}{\mathbf{n}^T\mathbf{n}}\mathbf{n} \\ &= \mathbf{x} - (\mathbf{n}^T\mathbf{x})\mathbf{n} \\ &= -A_{\mathbf{n}}^2\mathbf{x}, \end{aligned}$$

by Part b and the fact that $\mathbf{n}^T\mathbf{n} = \|\mathbf{n}\|^2 = 1$ (since $\mathbf{n}$ is a unit vector), as required.

4. First observe that

$$A^2 = \begin{pmatrix}2 & 1\\ c & d\end{pmatrix}\begin{pmatrix}2 & 1\\ c & d\end{pmatrix} = \begin{pmatrix}c+4 & d+2\\ c(d+2) & c+d^2\end{pmatrix}.$$

a. $A^2 = 0$ if, and only if, $c = -4$ and $d = -2$ (found by inspecting the entries in the first row of A^2, and verified by checking the entries in the second row of A^2).

b. $A^2 = I$ if, and only if, $c = -3$ and $d = -2$ (found by inspecting the entries in the first row of A^2, and verified by checking the entries in the second row of A^2).

c. A^2 is symmetric if, and only if, $c(d+2) = d+2$, or $(c-1)(d+2) = 0$, *i.e.*, if, and only if, $c = 1$ (and d is any real number) or $d = -2$ (and c is any real number).

5. a. Since

$$\begin{aligned} \begin{pmatrix}A & I_3\end{pmatrix} &= \begin{pmatrix}0 & 1 & 0 & 1 & 0 & 0\\ 3 & 0 & 0 & 0 & 1 & 0\\ 15 & 0 & 1 & 0 & 0 & 1\end{pmatrix} & \begin{matrix}R_1 \leftrightarrow R_2\\ R_3 \leftarrow R_3 - 5R_1\end{matrix} \\ &\sim \begin{pmatrix}3 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & -5 & 1\end{pmatrix} & R_1 \leftarrow \tfrac{1}{3}R_1 \\ &\sim \begin{pmatrix}1 & 0 & 0 & 0 & 1/3 & 0\\ 0 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & -5 & 1\end{pmatrix}, \end{aligned}$$

it follows that

$$A^{-1} = \begin{pmatrix}0 & 1/3 & 0\\ 1 & 0 & 0\\ 0 & -5 & 1\end{pmatrix}.$$

b. A is the product of the inverses of the elementary matrices corresponding to the elementary row operations used to reduce A to I_3, in the order of their application:

$$A = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 5 & 0 & 1\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

6. a. $A + C$ **might** be invertible. For example, if $A = I_n$ and C is the zero $n \times n$ matrix then $A + C$ is invertible. On the other hand, if $A = I_n$ and $C = \begin{pmatrix}-\mathbf{e}_1 & \mathbf{0} & \cdots & \mathbf{0}\end{pmatrix}$, then $A + C = \begin{pmatrix}\mathbf{0} & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{pmatrix}$ is singular.

b. A^TA **must** be invertible.

c. AC and BC **must** have the same determinant.

d. $\operatorname{Col} C$ **might** be equal to $\operatorname{Col} D$.

e. $\operatorname{Nul} A$ **must** be equal to $\operatorname{Nul} B$.

f. The columns of D **cannot** be linearly independent.

7. One has

$$\begin{pmatrix}3 & -1 & 3\\ 15 & -3 & 13\\ 12 & 2 & 10\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 5 & 1 & 0\\ 4 & 3 & 1\end{pmatrix}\begin{pmatrix}3 & -1 & 3\\ 0 & 2 & -2\\ 0 & 0 & 4\end{pmatrix}$$

via the rough work

$$\begin{matrix}2 & -2\\ 6 & -2\end{matrix} \rightsquigarrow 4.$$

8. a. $\det U = \det A = 2$, since $A = LU$ and $\det L = 1$;

b. $\det(3A^{-1}A^T) = 3^5(\det A)^{-1}(\det A) = 243$;

c. $\det(L + I) = 2^5 = 32$, because $L + I$ is a 5×5 triangular matrix each of whose diagonal entries is 2.

9. If $A\mathbf{x} = \mathbf{b}$ is the matrix equation corresponding to the linear system in question then (by adding row 4 to row 2, expanding along row 2 and then expanding along column 3)

$$\det A = \begin{vmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & 0 & 5 \\ -4 & 0 & 2 & 0 \\ 0 & -5 & 0 & -5 \end{vmatrix} = 15 \begin{vmatrix} 3 & -1 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 15(2) = 30,$$

so Cramer's Rule will give the value of x_2. Next, (expanding along row 2 and then expanding along row 3)

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 3 & 2 & -1 & 0 \\ 0 & 0 & 0 & 5 \\ -4 & -1 & 2 & 0 \\ 0 & 4 & 0 & -5 \end{vmatrix} = -20 \begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} = -40,$$

and so by Cramer's Rule $x_2 = -\frac{4}{3}$.

10. a. If p is the polynomial defined by $p(t) = 1$ then $p(0) = 1 \geqslant 0$, so $p \in \mathscr{S}_1$. However, $-p(0) = -1 \not\geqslant 0$, so $(-1)p = -p \notin \mathscr{S}_1$. Therefore, $\mathscr{S}_1$ is not a subspace of $\mathbb{P}_2$, since it is not closed under scalar multiplication.

b. Since $\mathscr{S}_2$ is the kernel of the linear transformation $T\colon M_{2\times 2} \to M_{2\times 2}$ defined by $T(X) = AX - XA$, it is a subspace of $M_{2\times 2}$. If

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

then

$$\begin{aligned} T(X) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_{21} + x_{12} & x_{22} - x_{11} \\ -x_{11} + x_{22} & -x_{12} - x_{21} \end{pmatrix} \end{aligned}$$

is the zero 2×2 matrix if, and only if, $x_{11} = x_{22}$ and $x_{12} = -x_{21}$; *i.e.*, $X = x_{22}I_2 + x_{12}A$. Therefore, $\mathscr{B} = \{I_2, A\}$ (since it is linearly independent and spans the kernel of T) is a basis of $\mathscr{S}_2$.

11. a. A direct calculation gives

$$T(p) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

b. If $p(t) = a_0 + a_1 t + a_2 t^2$ then $T(p) = \mathbf{0}$ if, and only if, $a_0 + a_1 + a_2 = 0$ and $a_0 + 2a_1 + 4a_2 = 0$. Reducing the coefficient matrix of this linear system,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix},$$

reveals that p belongs to the kernel of T if, and only if, $p(t) = a_2(2 - 3t + t^2)$, and that $\mathscr{B} = \{2 - 3t + t^2\}$ is a basis of the kernel of T.

12. a. Since the left and right factors in the factorization (where I denotes the 5×5 identity matrix)

$$B = \begin{pmatrix} A & A \\ A & A \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix},$$

are invertible, it follows that $\operatorname{rank}(B) = \operatorname{rank}(A) = 3$.

b. The rank formula implies that $\dim \operatorname{Nul}(B) = 10 - \operatorname{rank}(B) = 7$.

13. a. Points of intersection of ℓ_1 and ℓ_2 correspond to solutions of the equation $\mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$, or $s\mathbf{u} + t(-\mathbf{v}) = \mathbf{q} - \mathbf{p}$. Reducing the augmented matrix of the corresponding linear system,

$$\begin{pmatrix} \mathbf{u} & -\mathbf{v} & \mathbf{q} - \mathbf{p} \end{pmatrix} = \begin{pmatrix} 1 & -2 & 7 \\ 1 & 1 & 1 \\ 3 & -1 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$

gives $s = 3$, $t = -2$, and the point of intersection of ℓ_1 and ℓ_2,

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix},$$

is obtained by calculating $\mathbf{p} + 3\mathbf{u}$, or $\mathbf{q} - 2\mathbf{v}$.

b. The cosine of the angle between ℓ_1 and ℓ_2 is equal to

$$\frac{|\mathbf{u}^T\mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{4}{\sqrt{11}\sqrt{6}} = \tfrac{2}{33}\sqrt{66}.$$

c. The vector

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}$$

is orthogonal to the plane containing ℓ_1 and ℓ_2, which plane therefore has a normal equation $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{p}$, *i.e.*, $4x_1 + 5x_2 - 3x_3 = 17$.

14. a. The distance between the origin and the plane ϖ is the equal to length of the orthogonal projection of any $\mathbf{p} \in \varpi$ onto the given normal vector $\mathbf{n}$ to ϖ. This projection is

$$\mathbf{q} = \operatorname{proj}_{\mathbf{n}}(\mathbf{p}) = \frac{\mathbf{n}^T\mathbf{p}}{\mathbf{n}^T\mathbf{n}}\mathbf{n} = \tfrac{8}{9}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},$$

and so the distance between the origin and ϖ is $\|\mathbf{q}\| = \frac{8}{3}$.

b. The line which is orthogonal to ϖ and contains the origin is $\operatorname{Span}\{\mathbf{n}\}$ (has parametric vector equation $\mathbf{x} = t\mathbf{n}$), where $\mathbf{n}$ (from Part a) is the given normal vector to ϖ.

c. The point on ϖ which is closest to the origin is the point $\mathbf{q}$ found in Part a.

15. From $\operatorname{proj}_{\mathbf{w}}(\mathbf{u}) = \operatorname{proj}_{\mathbf{w}}(\mathbf{u} + \mathbf{v}) = \operatorname{proj}_{\mathbf{w}}(\mathbf{u}) + \operatorname{proj}_{\mathbf{w}}(\mathbf{v})$, it follows that $\operatorname{proj}_{\mathbf{w}}(\mathbf{v}) = \mathbf{0}$, and therefore that $\mathbf{v} = \operatorname{perp}_{\mathbf{w}}(\mathbf{v})$ is orthogonal to $\mathbf{w}$.

16. a. If $\mathbf{w} = \alpha\mathbf{e}_1$ then

$$\det\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix} = \begin{vmatrix} 2 & 1 & \alpha \\ 3 & -2 & 0 \\ 3 & 1 & 0 \end{vmatrix} = 9\alpha$$

must be equal to ± 10. So the required vectors are $\pm\frac{10}{9}\mathbf{e}_1$.

b. If $A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}$ and $B = \begin{pmatrix} \mathbf{a} & \mathbf{a} + \mathbf{b} & \mathbf{a} + \mathbf{b} + \mathbf{c} \end{pmatrix}$ then $B = AX$, where

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{and so} \qquad \det X = 1.$$

If it can be assumed that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are in $\mathbb{R}^3$ then the volume of the parallelepiped formed by the columns of B is $|\det B| = |\det A||\det X| = 6$. In any case, the square of the volume of the parallelepiped formed by the columns of B is $\det(B^TB) = \det(X^T)\det(A^TA)\det(X) = 36$, giving the same conclusion.

17. a. The statement is false; *e.g.*, let $\mathbf{a} = \mathbf{b} = \mathbf{e}_1$ and $\mathbf{u} = \mathbf{v} = \mathbf{e}_2$ in $\mathbb{R}^2$. Then $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are linearly dependent, but neither $\{\mathbf{a}, \mathbf{u}\}$ nor $\{\mathbf{a}, \mathbf{v}\}$ (each of which is $\{\mathbf{e}_1, \mathbf{e}_2\}$) are linearly dependent.

b. If α, β, γ and δ are scalars such that $\alpha\mathbf{a} + \beta\mathbf{u} = \mathbf{0}$ and $\gamma\mathbf{a} + \delta\mathbf{v} = \mathbf{0}$, then

$$\alpha\gamma\mathbf{a} + \beta\gamma\mathbf{u} = \mathbf{0}, \qquad \alpha\gamma\mathbf{a} + \alpha\delta\mathbf{v} = \mathbf{0} \qquad \text{and} \qquad \beta\gamma\mathbf{a} + \beta\delta\mathbf{v} = \mathbf{0}.$$

The first and second equations imply that $\beta\gamma\mathbf{u} - \alpha\delta\mathbf{v} = \mathbf{0}$, and so $\beta\gamma = 0$ and $\alpha\delta = 0$, since $\{\mathbf{u}, \mathbf{v}\}$, is linearly independent, and hence $\alpha\gamma = 0$ and $\beta\delta = 0$, since $\mathbf{a}$ and $\mathbf{v}$ are non-zero. Therefore, $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) = 0$; so either $\alpha = \beta = 0$, in which case $\{\mathbf{a}, \mathbf{u}\}$ is linearly independent, or $\gamma = \delta = 0$, in which case $\{\mathbf{a}, \mathbf{v}\}$ is linearly independent.

18. a.

$$A^2 = \begin{pmatrix} I & \mathbf{u} \\ \mathbf{u}^T & 0 \end{pmatrix}\begin{pmatrix} I & \mathbf{u} \\ \mathbf{u}^T & 0 \end{pmatrix} = \begin{pmatrix} I + \mathbf{u}\mathbf{u}^T & \mathbf{u} \\ \mathbf{u}^T & 1 \end{pmatrix},$$

since $\mathbf{u}^T\mathbf{u} = 1$ (because $\mathbf{u}$ is a unit vector).

b. Since

$$A = \begin{pmatrix} I & \mathbf{u} \\ \mathbf{u}^T & 0 \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{u}^T & 1 \end{pmatrix}\begin{pmatrix} I & \mathbf{u} \\ \mathbf{0}^T & -1 \end{pmatrix},$$

it follows that

$$A^{-1} = \begin{pmatrix} I & \mathbf{u} \\ \mathbf{0}^T & -1 \end{pmatrix}\begin{pmatrix} I & \mathbf{0} \\ -\mathbf{u}^T & 1 \end{pmatrix} = \begin{pmatrix} I - \mathbf{u}\mathbf{u}^T & \mathbf{u} \\ \mathbf{u}^T & -1 \end{pmatrix}.$$

c. Using the factorization from Part b gives

$$\det A = \det\begin{pmatrix} I & \mathbf{0} \\ \mathbf{u}^T & 1 \end{pmatrix}\det\begin{pmatrix} I & \mathbf{u} \\ \mathbf{0}^T & -1 \end{pmatrix} = (1)(-1) = -1.$$