1. Given

$$\begin{aligned} x_1 + 3x_2 - 10x_3 + 4x_4 &= 8 \\ x_2 - 4x_3 + 3x_4 &= 3 \\ x_1 + 4x_2 - 14x_3 + 7x_4 &= 11 \end{aligned}$$

a. Write the general solution of the system in parametric vector form.

b. Find a particular solution of the system in which $x_1 = x_2$ and $x_3 = x_4$.

2. In each part, give a 2×2 matrix which fits the description or explain why no such matrix exists.

a. A has all non-zero entries and $A^T = -A$.

b. A and $A + I$ are not invertible.

c. A is the standard matrix of a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is injective but not surjective.

3. Given the matrix equation

$$\begin{pmatrix} 5 & 0 & 7 \\ 1 & 2 & 2 \\ 4 & \alpha & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix}.$$

a. Use Cramer's Rule to solve for x_2 only, in terms of α.

b. Are there any values of α for which the equation has infinitely many solutions?

4. Find a polynomial p of degree at most 2 such that $p(2) = 3$, $p(-3) = 8$ and $p'(-1) = -3$.

5. Let A, C and D be 3×3 matrices such that $\det A = -2$, C is invertible and D is not invertible. Find the value of:

a. $\det(3A)$; b. $\det(ACA) - \det(CA^2)$;

c. $\det(C^T C^{-1})$; d. $\det(AD + CD)$.

6. Find A^{-1}, and use it to solve $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & -2 & -4 \\ -3 & 3 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}.$$

7. Write an LU factorization of the matrix $\begin{pmatrix} 2 & 2 & 1 & 2 \\ 6 & 9 & 7 & 9 \\ -8 & 7 & 15 & 7 \end{pmatrix}$.

8. Upon a 3×3 matrix A you perform the elementary row operations (in order).

- Multiply the first row of A by 5.
- Add 4 times the third row to the second row.
- Interchange the first and third row.

The resulting matrix is I_3.

a. Write A^{-1} and A as products of elementary matrices.

b. What is $\det A$?

9. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the composite (in the given order) of:

- a rotation about the origin through $\frac{1}{4}\pi$ (radians, counterclockwise);
- a vertical expansion by a factor of 2;
- a reflection in the x_2 axis;

a. Find the standard matrix of T.

b. Let $\mathscr{D} = \{\, \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leqslant 1 \,\}$. What is the area of the image of $\mathscr{D}$ under T?

c. Sketch the image of $\mathscr{D}$ under T.

10. Let $U = \begin{pmatrix} 0 & I \\ A & C \end{pmatrix}$.

a. Write U^{-1} as a partitioned matrix.

b. Use Part a to find the inverse of

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 2 & 1 & -2 & -3 \end{pmatrix}.$$

11. Let $\mathscr{H} = \{\, A \in M_{2\times 2} : A^2 = 0 \,\}$.

a. Does $\mathscr{H}$ contain the zero matrix?

b. Give three non-zero elements of $\mathscr{H}$.

c. Is $\mathscr{H}$ closed under addition? Justify your answer.

d. Is $\mathscr{H}$ closed under scalar multiplication? Justify your answer.

e. Is $\mathscr{H}$ a subspace of $M_{2\times 2}$? Justify your answer.

f. Give a non-zero matrix $A \in \mathscr{H}$ such that $\operatorname{Nul}(A) \neq \operatorname{Col}(A)$, or explain why there is no such matrix.

12. Let V be a vector space, and let $\mathbf{u}$, $\mathbf{v}$ be linearly independent vectors in V.

a. Show that $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

b. Give an explicit dependence relation satisfied by $\mathbf{u} + \mathbf{v}$, $2\mathbf{u} - \mathbf{v}$, $\mathbf{u}$.

13. Explain why $\mathscr{X} = \{\, p \in \mathbb{P}_3 : p(1) = 0,\ p'(1) = 0 \,\}$ is a subspace of $\mathbb{P}_3$, and give a basis of $\mathscr{X}$.

14. Let $\wp_1$ and $\wp_2$ be the planes in $\mathbb{R}^3$ defined, respectively, by

$$x + 2y + 3z = 7 \qquad \text{and} \qquad 2x + 5y + 6z = 19,$$

let A be the point $(1, 1, 1)$, and let ℓ be the line of intersection of $\wp_1$ and $\wp_2$.

a. Write a parametric vector equation of the line ℓ.

b. Give a parametric vector equation of the line perpendicular to $\wp_2$ through A.

c. Find the distance between A and $\wp_2$.

d. Find the cosine of the angle between $\wp_1$ and $\wp_2$.

15. Let

$$\mathbf{u} = \begin{pmatrix} 4 \\ -2 \\ k \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For each of the following, find all values of k such that

a. $\mathbf{u}$ is perpendicular to $\mathbf{v}$.

b. $\mathbf{u}$ is parallel to $\mathbf{v}$.

c. $\|\mathbf{u}\| = \|\mathbf{v}\|$.

d. The area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{w}$ is 5.

e. $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \mathbf{v}$.

16. a. Show that if A is an invertible $m \times m$ matrix and B is an $m \times n$ matrix then $\operatorname{Nul}(AB) = \operatorname{Nul}(B)$.

b. Prove that if A and B are 3×3 matrices of rank 1 then there is a non-zero vector in $\operatorname{Nul}(A) \cap \operatorname{Nul}(B)$.

1. a. Let $A = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{b}\end{pmatrix}$ be the augmented matrix of the given linear system. By inspection (the second entry of $\mathbf{a}_2$ gives everything away), $\mathbf{a}_1$, $\mathbf{a}_2$ are linearly independent, $\mathbf{a}_3 = 2\mathbf{a}_1 - 4\mathbf{a}_2$, $\mathbf{a}_4 = -5\mathbf{a}_1 + 3\mathbf{a}_2$ and $\mathbf{b} = -\mathbf{a}_1 + 3\mathbf{a}_2$, so the general solution of the linear system is $\mathbf{p} + \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{p} = \begin{pmatrix}-1\\3\\0\\0\end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix}-2\\4\\1\\0\end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix}5\\-3\\0\\1\end{pmatrix}.$$

b. The required solution has the form $\mathbf{p} + s(\mathbf{u} + \mathbf{v})$ where $-1 + 3s = 3 + s$, *i.e.*, $s = 2$. So $\mathbf{p} + 2(\mathbf{u} + \mathbf{v})$ is the required solution.

2. a. $A^T = -A$ implies that the diagonal entries of A are zero, so no such matrix is possible.

b. Neither $A = \begin{pmatrix}-\mathbf{e}_1 & \mathbf{0}\end{pmatrix}$ nor $A + I = \begin{pmatrix}\mathbf{0} & \mathbf{e}_2\end{pmatrix}$ is invertible.

c. If the linear transformation $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is injective then trivially A is invertible, so there is no such matrix.

3. a. If A is the coefficient matrix, and $\mathbf{b}$ the right hand side, of the given equation, then

$$\det(A) = \begin{vmatrix}5 & 0 & 7\\1 & 2 & 2\\4 & \alpha & 5\end{vmatrix} = 2\begin{vmatrix}5 & 7\\4 & 5\end{vmatrix} - \alpha\begin{vmatrix}5 & 7\\1 & 2\end{vmatrix} = -3(\alpha + 2),$$

so Cramer's Rule can be applied if $\alpha \neq -2$. Since

$$\det A_2(\mathbf{b}) = \begin{vmatrix}5 & 1 & 7\\1 & \alpha & 2\\4 & 1 & 5\end{vmatrix} = \begin{vmatrix}0 & 1-5\alpha & -3\\1 & \alpha & 2\\0 & 1-4\alpha & -3\end{vmatrix} = 3\begin{vmatrix}1-5\alpha & 1\\1-4\alpha & 1\end{vmatrix} = -3\alpha,$$

$x_2 = \alpha/(\alpha + 2)$ provided $\alpha \neq -2$.

b. The equation has a unique solution unless $\alpha = -2$, in which case the rank of A is 2 and $\operatorname{rank}\begin{pmatrix}A & \mathbf{b}\end{pmatrix} \geqslant \operatorname{rank} A_2(\mathbf{b}) = 3$, which implies that $A\mathbf{x} = \mathbf{b}$ is inconsistent. So there are no values of α for which the equation has infinitely many solutions.

4. The coefficients of a polynomial $p(x) = a_0 + a_1x + a_2x^2$ are solutions of the linear system whose augmented matrix is

$$\begin{pmatrix}1 & 2 & 4 & 3\\1 & -3 & 9 & 8\\0 & 1 & -2 & -3\end{pmatrix} \qquad R_2 \leftarrow R_2 - R_1$$

$$\sim \begin{pmatrix}1 & 2 & 4 & 3\\0 & -5 & 5 & 5\\0 & 1 & -2 & -3\end{pmatrix} \qquad R_3 \leftarrow R_3 + \tfrac{1}{5}R_2$$

$$\sim \begin{pmatrix}1 & 2 & 4 & 3\\0 & -5 & 5 & 5\\0 & 0 & -1 & -2\end{pmatrix} \qquad \begin{matrix}R_1 \leftarrow R_1 + 4R_3\\R_2 \leftarrow R_1 + 5R_3\\R_3 \leftarrow -R_3\end{matrix}$$

$$\sim \begin{pmatrix}1 & 2 & 0 & -5\\0 & -5 & 0 & -5\\0 & 0 & 1 & 2\end{pmatrix} \qquad \begin{matrix}R_1 \leftarrow R_1 + \tfrac{2}{5}R_2\\R_2 \leftarrow -\tfrac{1}{5}R_2\end{matrix}$$

$$\sim \begin{pmatrix}1 & 0 & 0 & -7\\0 & 1 & 0 & 1\\0 & 0 & 1 & 2\end{pmatrix}.$$

So $p(x) = -7 + x + 2x^2$ satisfies $p(2) = 3$, $p(-3) = 8$ and $p'(-1) = -3$.

5. Since the determinant is multilinear, preserves products and is invariant under transposition,

a. $\det(3A) = 3^3(\det A) = -54$,

b. $\det(ACA) - \det(CA^2) = (\det A)(\det C)(\det A) - (\det C)(\det A)^2 = 0$,

c. $\det(C^TC^{-1}) = (\det C)(\det C)^{-1} = 1$ and

d. $\det(AD + CD) = \det(A + C)\,(\det D) = 0$.

6. a. Reducing $\begin{pmatrix}A & I_3\end{pmatrix}$ gives

$$\begin{pmatrix}0 & 1 & 3 & 1 & 0 & 0\\1 & -2 & -4 & 0 & 1 & 0\\-3 & 3 & 4 & 0 & 0 & 1\end{pmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix}1 & -2 & -4 & 0 & 1 & 0\\0 & 1 & 3 & 1 & 0 & 0\\-3 & 3 & 4 & 0 & 0 & 1\end{pmatrix} \qquad R_3 \leftarrow R_3 + 3R_1$$

$$\sim \begin{pmatrix}1 & -2 & -4 & 0 & 1 & 0\\0 & 1 & 3 & 1 & 0 & 0\\0 & -3 & -8 & 0 & 3 & 1\end{pmatrix} \qquad R_3 \leftarrow R_3 + 3R_2$$

$$\sim \begin{pmatrix}1 & -2 & -4 & 0 & 1 & 0\\0 & 1 & 3 & 1 & 0 & 0\\0 & 0 & 1 & 3 & 3 & 1\end{pmatrix} \qquad \begin{matrix}R_1 \leftarrow R_1 + 4R_3\\R_2 \leftarrow R_2 - 3R_3\end{matrix}$$

$$\sim \begin{pmatrix}1 & -2 & 0 & 12 & 13 & 4\\0 & 1 & 0 & -8 & -9 & -3\\0 & 0 & 1 & 3 & 3 & 1\end{pmatrix} \qquad R_1 \leftarrow R_1 + 2R_2$$

$$\sim \begin{pmatrix}1 & 0 & 0 & -4 & -5 & -2\\0 & 1 & 0 & -8 & -9 & -3\\0 & 0 & 1 & 3 & 3 & 1\end{pmatrix},$$

and so

$$A^{-1} = \begin{pmatrix}-4 & -5 & -2\\-8 & -9 & -3\\3 & 3 & 1\end{pmatrix}.$$

b. From Part a one has

$$\mathbf{x} = \begin{pmatrix}-4 & -5 & -2\\-8 & -9 & -3\\3 & 3 & 1\end{pmatrix}\begin{pmatrix}4\\-5\\1\end{pmatrix} = \begin{pmatrix}7\\10\\-2\end{pmatrix}.$$

7. One has

$$\begin{pmatrix}2 & 2 & 1 & 2\\6 & 9 & 7 & 9\\-8 & 7 & 15 & 7\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\3 & 1 & 0\\-4 & 5 & 1\end{pmatrix}\begin{pmatrix}2 & 2 & 1 & 2\\0 & 3 & 4 & 3\\0 & 0 & -1 & 0\end{pmatrix},$$

via the rough work

$$\begin{matrix}3 & 4 & 3\\15 & 19 & 15\end{matrix} \quad \rightsquigarrow \quad -1 \quad 0.$$

8. It is given that $E_3E_2E_1A = I_3$, and so

a.

$$A^{-1} = E_3E_2E_1 = \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 4\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}5 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \qquad \text{and}$$

b.

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix}\frac{1}{5} & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & -4\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}.$$

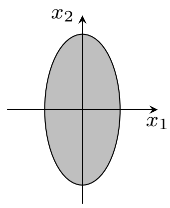
c. The determinant of A is the product of the determinants of the matrices in its elementary factorization from Part b, which is $-\frac{1}{5}$.

9. a. The standard matrix A of T is the product of the standard matrices of the given transformations, *i.e.*,

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} = \tfrac{1}{2}\sqrt{2} \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$$

b. The area of the image of $\mathscr{D}$ under T is $|\det A|\pi = 2\pi$.

c. The image of $\mathscr{D}$ under the action of T is displayed below; the intercepts of the ellipse are $\pm\mathbf{e}_1$ and $\pm 2\mathbf{e}_2$.



10. a. Since

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & C \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} R & S \\ AP + CR & AQ + CS \end{pmatrix}$$

implies that $R = I$, $S = 0$, and therefore $AP + C = 0$, or $P = -A^{-1}C$, and $AQ = I$, or $Q = A^{-1}$, it follows that

$$\begin{pmatrix} 0 & I \\ A & C \end{pmatrix}^{-1} = \begin{pmatrix} -A^{-1}C & A^{-1} \\ I & 0 \end{pmatrix}.$$

b. The matrix M is partitioned as in Part a with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad \text{and} \quad C = -\begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 2 & 3 \end{pmatrix},$$

so

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad -A^{-1}C = \begin{pmatrix} 2 & 3 \\ 2 & 3 \\ -2 & -3 \end{pmatrix};$$

therefore,

$$M^{-1} = \begin{pmatrix} 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ -2 & -3 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

11. a. Since $0_{2\times 2}^2 = 0_{2\times 2}$, $\mathscr{H}$ does contain the 2×2 zero matrix.

b. The linear transformations $T_j \colon \mathbb{R}^2 \to \mathbb{R}^2$, determined by $T_1(\mathbf{e}_1) = \mathbf{e}_2$, $T_1(\mathbf{e}_2) = \mathbf{0}$, $T_2(\mathbf{e}_1) = \mathbf{0}$, $T_2(\mathbf{e}_2) = \mathbf{e}_1$ and $T_3(\mathbf{e}_1) = T_3(\mathbf{e}_2) = \mathbf{e}_1 - \mathbf{e}_2$, satisfy $T_j \circ T_j = 0$ (the zero transformation $\mathbb{R}^2 \to \mathbb{R}^2$) for $1 \leqslant j \leqslant 3$, and so

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

(their standard matrices) are non-zero matrices belonging to $\mathscr{H}$.

c. By Parts b

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A^T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

belong to $\mathscr{H}$, but

$$A + A^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and so} \quad (A + A^T)^2 = I_2,$$

so $A + A^T$ does not belong to $\mathscr{H}$. Therefore, $\mathscr{H}$ is not closed under addition.

d. If $A \in \mathscr{H}$ and $\alpha \in \mathbb{R}$, then $A^2 = 0_{2\times 2}$, and so

$$(\alpha A)^2 = \alpha^2 A^2 = \alpha^2 0_{2\times 2} = 0_{2\times 2};$$

hence, $\alpha A \in \mathscr{H}$. Therefore, $\mathscr{H}$ is closed under scalar multiplication.

e. Since $\mathscr{H}$ is not closed under addition, it is not a subspace of $M_{2\times 2}$.

f. If $A \in \mathscr{H}$ and $\mathbf{y} \in \operatorname{Col}(A)$ then $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^2$, and therefore $A\mathbf{y} = A^2\mathbf{x} = 0\mathbf{x} = \mathbf{0}$, which implies that $\mathbf{y} \in \operatorname{Nul}(A)$. This shows that if $A \in \mathscr{H}$ then $\operatorname{Col}(A)$ is a subspace of $\operatorname{Nul}(A)$. So if $A \in \mathscr{H}$ and $A \neq 0$ then $1 \leqslant \dim \operatorname{Col}(A) \leqslant \dim \operatorname{Nul}(A) \leqslant 1$; *i.e.*, $\dim \operatorname{Col}(A) = \dim \operatorname{Nul}(A)$, and therefore $\operatorname{Col}(A) = \operatorname{Nul}(A)$. Hence, there is no such matrix.

12. It is given that $W = \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$ is a 2-dimensional linear space.

a. Since $\mathbf{u} = \frac{1}{2}(\mathbf{u}+\mathbf{v}) + \frac{1}{2}(\mathbf{u}-\mathbf{v})$ and $\mathbf{v} = \frac{1}{2}(\mathbf{u}+\mathbf{v}) - \frac{1}{2}(\mathbf{u}-\mathbf{v})$, $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ is a list of two vectors which spans a 2-dimensional linear space, and is therefore linearly independent.

b. By inspection, $(\mathbf{u}+\mathbf{v}) + (2\mathbf{u}-\mathbf{v}) - 3\mathbf{u} = \mathbf{0}$, is a (non-trivial) linear dependence equation satisfied by the given vectors.

13. The set $\mathscr{X}$ is the kernel of the linear transformation $T \colon \mathbb{P}_3 \to \mathbb{R}^2$ defined by

$$T(p) = \begin{pmatrix} p(1) \\ p'(1) \end{pmatrix},$$

so it is a subspace of $\mathbb{P}_3$. The matrix of T relative to the bases, $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$, and $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$, is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad \text{so} \quad \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

are the coordinate vectors of polynomials which form a basis of the kernel of T; *i.e.*, $\{1 - 2t + t^2, 2 - 3t + t^3\}$ is a basis of the kernel of T.

14. a. Reducing the augmented matrix of the system of the equations of $\wp_1$ and $\wp_2$ gives

$$\begin{pmatrix} 1 & 2 & 3 & 7 \\ 2 & 5 & 6 & 19 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & 0 & 5 \end{pmatrix},$$

so the line ℓ of intersection of ϖ_1 and ϖ_2 is $\mathbf{p} + \operatorname{Span}\{\mathbf{u}\}$, where

$$\mathbf{p} = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

b. The line in question is $\mathbf{a} + \operatorname{Span}\{\mathbf{n}_2\}$ where $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$$

is the given normal vector to the plane ϖ_2.

c. The distance between A and ϖ_2 is the length of the projection of $\mathbf{a} - \mathbf{q}$ onto $\mathbf{n}_2$, where $\mathbf{q}$ is the coordinate vector of any point on ϖ_2 and $\mathbf{n}_2$ is a normal vector to ϖ_2. The length of this projection is equal to

$$\frac{|\mathbf{n}_2^T\mathbf{a} - 19|}{\|\mathbf{n}\|} = \frac{|13 - 19|}{\sqrt{65}} = \tfrac{6}{65}\sqrt{65}.$$

d. The cosine of the angle between ϖ_1 and ϖ_2 is equal to the absolute value of the cosine of the angle between their given normal vectors, which is

$$\frac{|\mathbf{n}_1^T\mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{30}{\sqrt{14}\sqrt{65}} = \tfrac{3}{91}\sqrt{910}$$

15. a. $\mathbf{u}$ is perpendicular to $\mathbf{v}$ if, and only if, $0 = \mathbf{u}^T\mathbf{v} = -30 + k$, *i.e.*, $k = 30$.

b. $\mathbf{u}$ is parallel to $\mathbf{v}$ if, and only if, $\mathbf{u} = -\frac{2}{3}\mathbf{v}$, *i.e.*, $k = -\frac{2}{3}$.

c. $\|\mathbf{u}\| = \|\mathbf{v}\|$ if, and only if, $\mathbf{u}^T\mathbf{u} = \mathbf{v}^T\mathbf{v}$, *i.e.*, $20 + k^2 = 46$, or $k = \pm\sqrt{26}$.

d. The area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{w}$ is 5 if, and only if, the length of $\mathbf{u} \times \mathbf{w}$ is 5, *i.e.*, $25 = (\mathbf{u} \times \mathbf{w})^T(\mathbf{u} \times \mathbf{w}) = k^2 + 16$, or $k = \pm 3$.

e. $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \mathbf{v}$ if, and only if, $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{v}$, *i.e.*, $-30 + k = 46$, or $k = 76$.

16. a. If $\mathbf{x} \in \operatorname{Nul} B$ then $B\mathbf{x} = \mathbf{0}$, and so $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$, which means that $\mathbf{x} \in \operatorname{Nul}(AB)$. Therefore $\operatorname{Nul}(B) \subset \operatorname{Nul}(AB)$. Conversely, if $\mathbf{x} \in \operatorname{Nul}(AB)$ then $AB\mathbf{x} = \mathbf{0}$, and so $B\mathbf{x} = A^{-1}AB\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$, which means that $\mathbf{x} \in \operatorname{Nul} B$. Hence, $\operatorname{Nul}(AB) \subset \operatorname{Nul}(B)$, and therefore $\operatorname{Nul}(AB) = \operatorname{Nul}(B)$.

b. If $\operatorname{rank}(A) = \operatorname{rank}(B) = 1$, then $\dim \operatorname{Nul}(A) = \dim \operatorname{Nul}(B) = 2$ (since A and B are 3×3 matrices), and so

$$\begin{aligned} 3 &\geqslant \dim\big(\operatorname{Nul}(A) + \operatorname{Nul}(B)\big) \\ &= \dim \operatorname{Nul}(A) + \dim \operatorname{Nul}(B) - \dim\big(\operatorname{Nul}(A) \cap \operatorname{Nul}(B)\big) \\ &= 4 - \dim\big(\operatorname{Nul}(A) \cap \operatorname{Nul}(B)\big), \end{aligned}$$

and therefore, $\dim\big(\operatorname{Nul}(A) \cap \operatorname{Nul}(B)\big) \geqslant 1$, which implies that there is a non-zero vector in $\operatorname{Nul}(A) \cap \operatorname{Nul}(B)$.