1. Solve the system and give a basis for the column space of its coefficient matrix.

$$\begin{aligned} x_1 + 2x_2 + 6x_3 + 4x_4 &= 5 \\ x_2 + 5x_3 + 3x_4 &= 7 \\ x_1 + 3x_2 + 11x_3 + 7x_4 &= 12 \end{aligned}$$

2. In each part, give a 3×3 matrix A which fits the description or explain why there is no such matrix.

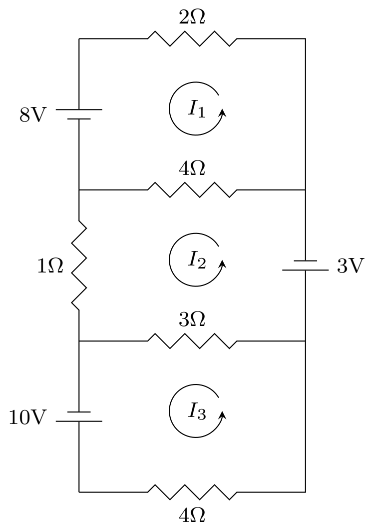
a. $\dim\big(\mathrm{Nul}(A)\big) = 0$.

b. The columns of A are linearly dependent but the rows of A are linearly independent.

c. The null space of A is a plane.

d. A has rank 1 and $I + A$ is singular.

3. Write the augmented matrix of a linear system whose solution gives the loop currents in the circuit below. Do not solve the linear system.



4. Let A, B and C be 4×4 such that $\det(A) = -2$, $\det(B) = 3$ and C is not invertible. Find the value of each of the following.

a. $\det(-5A^2B^{-1})$ b. $\det(\mathrm{adj}\, B)$ c. $\det\big((ABC)^T\big)$

5. Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 6 & -1 \\ 1 & 2 & -3 \\ 3 & 7 & -6 \end{pmatrix}.$$

6. Write an LU factorization of the matrix

$$A = \begin{pmatrix} 2 & 5 & 4 \\ 6 & 12 & 6 \\ -4 & -22 & -27 \end{pmatrix}.$$

7. Use Cramer's Rule to solve the following equation for x_2 only.

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

8. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation which shifts the first three entries down one spot and brings the negative of the last entry to the top. For example,

$$T \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

a. Find the standard matrix A of T. b. Find A^{87}.

9. A non-zero square matrix is said to be *nilpotent of degree* 2 if $A^2 = 0$.

a. Give an example of a 2×2 matrix which is nilpotent of degree 2.

b. Show that if A is nilpotent of degree 2 then so is the partitioned matrix

$$\begin{pmatrix} A & 0 \\ I & -A \end{pmatrix}.$$

c. Suppose that A is an $n \times n$ matrix which is nilpotent of degree 2. Is there a non-zero scalar k such that $A + kI$ is nilpotent of degree 2?

10. Let

$$\mathscr{H} = \{\, A \in M_{2\times 2} \colon A\mathbf{v} = \mathbf{0} \,\}, \qquad \text{where} \qquad \mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

a. Find a non-zero matrix in $\mathscr{H}$.

b. Does $\mathscr{H}$ contain the zero matrix? Justify.

c. Is $\mathscr{H}$ closed under addition? Justify.

d. Is $\mathscr{H}$ closed under scalar multiplication? Justify.

e. Is $\mathscr{H}$ a subspace of $M_{2\times 2}$? Justify.

11. Find a basis of the vector space

$$\mathscr{V} = \{\, \mathbf{p}(t) \in \mathbb{P}_3 \colon \mathbf{p}(-2) = 0, \text{ and } \mathbf{p}(2) = 0 \,\}.$$

12. You are given $A(-3, 1, -2)$, and the plane $\mathscr{P}$ defined by $x - 2y - 3z = -4$.

a. Give a parametric vector equation of the line through A which is perpendicular to $\mathscr{P}$.

b. Find the point on $\mathscr{P}$ which is closest to A.

c. Find an equation of the form $ax + by + cz = d$ of the plane which contains A and is parallel to $\mathscr{P}$.

d. What is the distance between A and $\mathscr{P}$?

e. The plane $-4x + 7y + kz = h$ is perpendicular to $\mathscr{P}$ and contains A. Find k and h.

13. Let

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

a. Find a unit vector $\mathbf{u}$ which is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

b. Find the volume of the parallelepiped $\mathscr{P}$ formed by $\mathbf{v}$, $\mathbf{w}$ and the vector $\mathbf{u}$ from Part a.

c. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with standard matrix

$$A = \begin{pmatrix} 3 & 2 & 9 \\ 0 & -4 & 3 \\ 0 & 0 & 5 \end{pmatrix}.$$

Find the volume of $T(\mathscr{P})$; that is, the image of $\mathscr{P}$ under T.

14. Find a condition on a, b and c so that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{is in the span of} \quad \left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 14 \\ 6 \\ 4 \end{pmatrix} \right\}.$$

15. Let A, B and C be invertible matrices such that $B^{-1}AB + B^{-1}C = I$.

a. Solve for A in terms of the other matrices.

b. Prove that B cannot equal C.

16. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation with standard matrix A.

a. If $m > n$, find an expression for the maximum possible value of $\dim(\mathrm{Col}\, A)$.

b. If $m > n$, is it possible for T to be injective (one-to-one)? Justify.

c. If $m = 4$ and $n = 6$ and $\dim(\mathrm{Nul}\, A) = 3$, give the dimension of the column space, row space and null space of A^T.

17. Complete each of the following sentences with **must**, **might** or **cannot**.

a. If $\mathbf{x} \in \mathrm{Nul}(A)$, then $-2\mathbf{x}$ _____ also be in $\mathrm{Nul}(A)$.

b. If $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w}$ _____ be orthogonal to $\mathbf{u} + \mathbf{v}$.

c. If $\mathbf{u}$ is parallel to $\mathbf{x}$ and $\mathbf{v}$ is parallel to $\mathbf{y}$ then $\mathbf{u} + \mathbf{v}$ _____ be parallel to $\mathbf{x} + \mathbf{y}$.

d. If E_1, E_2 are elementary matrices, then E_1E_2 _____ be an elementary matrix.

18. Let $T\colon V \to W$ be an injective (one-to-one) linear transformation of vector spaces. Show that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

1. The system is equivalent to a matrix equation $\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4\end{pmatrix}\mathbf{x} = \mathbf{b}$, and by inspection, $\mathbf{a}_3 = -4\mathbf{a}_1 + 5\mathbf{a}_2$, $\mathbf{a}_4 = -2\mathbf{a}_1 + 3\mathbf{a}_2$ and $\mathbf{b} = -9\mathbf{a}_1 + 7\mathbf{a}_2$, so $-9\mathbf{e}_1 + 7\mathbf{e}_2 + \operatorname{Span}\{4\mathbf{e}_1 - 5\mathbf{e}_2 + \mathbf{e}_3, 2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_4\}$ is the solution set of the system, and $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of the column space of its coefficient matrix.

2. I_3 is an example for Part a., there is no example for Part b because rank is invariant under transposition, and $\begin{pmatrix}-\mathbf{e}_1 & \mathbf{0} & \mathbf{0}\end{pmatrix}$ is an example as for Parts c and d.

3. Ohm's and Kirchhoff's laws give

$$\begin{pmatrix} 6 & -4 & 0 & -8 \\ -4 & 8 & -3 & -3 \\ 0 & -3 & 7 & 10 \end{pmatrix}.$$

4. Since the determinant is multilinear, preserves products and is invariant under transposition, it follows that
a. $\det(-5A^2B^{-1}) = (-5)^4(-2)^2/3 = 2500/3$,
b. $\det(\operatorname{adj} B) = \det\big((\det B)B^{-1}\big) = 3^4/3 = 27$ and
c. $\det\big((ABC)^T\big) = (-2)(3)(0) = 0$.

5. Reducing A to I_3 (using the row reduction algorithm)

$$A \sim \begin{pmatrix} 2 & 6 & -1 \\ 0 & -1 & -\frac{5}{2} \\ 0 & -2 & -\frac{9}{2} \end{pmatrix} \sim \begin{pmatrix} 2 & 6 & -1 \\ 0 & -1 & -\frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 2 & 6 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim I_3,$$

and applying the same sequence of row operations to I_3,

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -4 & 2 \\ -3 & -9 & 5 \\ -1 & -4 & 2 \end{pmatrix} \sim \begin{pmatrix} -9 & -29 & 16 \\ 3 & 9 & -5 \\ -1 & -4 & 2 \end{pmatrix},$$

yields the inverse of A.

6. An LU factorization of A is

$$\begin{pmatrix} 2 & 5 & 4 \\ 6 & 12 & 6 \\ -4 & -22 & -27 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 & 4 \\ 0 & -3 & -6 \\ 0 & 0 & 5 \end{pmatrix},$$

via the rough work

$$\begin{matrix} -3 & -6 \\ -12 & -19 \end{matrix} \rightsquigarrow 5.$$

7. Since

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 7 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 7 & 1 \\ 1 & 3 \end{vmatrix} = -20,$$

and

$$\begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 9,$$

$x_2 = -\frac{9}{20}$.

8. a. The standard matrix of T is

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) & T(\mathbf{e}_4) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & -\mathbf{e}_1 \end{pmatrix}.$$

b. Since $T^4 = -\operatorname{id}_{\mathbb{R}^4}$ and T maps the standard basis of $\mathbb{R}^4$ to a list of orthogonal unit vectors, it follows that $A^{87} = A^{-1} = A^T = \begin{pmatrix} -\mathbf{e}_4 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix}$.

9. a. The matrix $\begin{pmatrix}\mathbf{e}_2 & \mathbf{0}\end{pmatrix}$ is nilpotent of degree 2.
b. As the sum of two anti-commuting matrices which are nilpotent of degree 2,

$$\begin{pmatrix} A & 0 \\ I & -A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$

is nilpotent of degree 2.
c. If $(A + kI)^2 = 0$ and $k \neq 0$ then A is invertible (since A satisfies a polynomial equation with non-zero constant term), so there is no such scalar.

10. $\begin{pmatrix}-2\mathbf{e}_1 & 5\mathbf{e}_1\end{pmatrix} \in H$ is non-zero, and H is a subspace of $M_{2\times 2}$ since it is the kernel of the linear map $T\colon M_{2\times 2} \to \mathbb{R}^2$, defined by $T(A) = A(5\mathbf{e}_1 + 2\mathbf{e}_2)$.

11. Since V is the kernel of the linear transformation $T\colon \mathbb{P}_3 \to \mathbb{R}^2$ defined by $T(\mathbf{p}) = \mathbf{p}(2)\mathbf{e}_1 + \mathbf{p}(-2)\mathbf{e}_2$, and $\operatorname{rank}(T) = 2$, any linearly independent pair of multiples of $4 - t^2$ is a basis of V; *e.g.*, $\{4 - t^2, t(4 - t^2)\}$.

12. a. The line containing $\mathbf{a}$ and orthogonal to $\mathscr{P}$ is $\mathbf{a} + \operatorname{Span}\{\mathbf{n}\}$, where $\mathbf{n}$ is the given normal vector to $\mathscr{P}$.
b. The point on $\mathscr{P}$ which is closest to $\mathbf{a}$ is $\mathbf{a} + \operatorname{proj}_{\mathbf{n}}(\mathbf{p} - \mathbf{a})$, *i.e.*,

$$\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} - \frac{5}{14}\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -47 \\ 24 \\ -13 \end{pmatrix},$$

where $\mathbf{p}$ is any point on $\mathscr{P}$.
c. The plane which contains $\mathbf{a}$ and is parallel to $\mathscr{P}$ is defined by $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{a}$, or $x_1 - 2x_2 - 3x_3 = 1$.
d. From Part b, the distance between $\mathbf{a}$ and $\mathscr{P}$ is $\|\operatorname{proj}_{\mathbf{n}}(\mathbf{p} - \mathbf{a})\| = \frac{5}{14}\sqrt{14}$.
e. If the plane $\mathscr{P}'$ is orthogonal to $\mathscr{P}$, then $0 = \mathbf{n}^T\mathbf{n}' = -18 - 3k$, so $k = -6$, and $h = \mathbf{n}'^T\mathbf{a} = 31$, where $\mathbf{n}'$ is the given normal vector to $\mathscr{P}'$.

13. a. A unit vector orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ is the normalization of

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}, \quad \textit{i.e.}, \quad \frac{1}{26}\sqrt{26}\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

b. The volume of $\mathscr{P}$ is equal to the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, *i.e.*, $\|\mathbf{v} \times \mathbf{w}\| = \sqrt{26}$.
c. The volume of $T(\mathscr{P})$ is $|\det(T)|\sqrt{26} = 60\sqrt{26}$.

14. The given set is the orthogonal complement of the null space of the matrix

$$\begin{pmatrix} 2 & 1 & 2 \\ -3 & -1 & 2 \\ 14 & 6 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1/2 & 5 \\ 0 & -1 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 10 \\ 0 & 0 & 0 \end{pmatrix},$$

so it is the general solution of $4a - 10b + c = 0$.

15. a. The given equation is equivalent to $B^{-1}AB = I - B^{-1}C$, and therefore also to $A = I - CB^{-1}$.
b. If $B = C$ then A is invertible and $A = 0$, which is absurd.

16. a. If $m > n$ then the dimension of $\operatorname{Col}(A)$ can be n but no larger.
b. If $m > n$ then there are many possible linear injections of $\mathbb{R}^n$ into $\mathbb{R}^m$.
c. If $m = 4$, $n = 6$ and the dimension of the null space of A is 3, then the rank of A, which is the dimension of the column spaces of A and of A^T, is $6 - 3 = 3$, and the dimension of the null space of A^T is is $4 - 3 = 1$.

17. a. If $\mathbf{x} \in \operatorname{Nul}(A)$, then $-2\mathbf{x}$ **must** be in $\operatorname{Nul}(A)$.
b. If $\mathbf{w}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w}$ **must** be orthogonal to $\mathbf{u} + \mathbf{v}$.
c. If $\mathbf{u}$ is parallel to $\mathbf{x}$ and $\mathbf{v}$ is parallel to $\mathbf{y}$, then $\mathbf{u} + \mathbf{v}$ **might** be parallel to $\mathbf{x} + \mathbf{y}$.
d. If E_1, E_2 are elementary matrices, then E_1E_2 **might** be an elementary matrix.

18. If $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent then there are scalars α_1, α_2 and α_3, not all zero, such that

$$\alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \alpha_3 T(\mathbf{v}_3) = \mathbf{0}_W.$$

Since T is linear, this equation is equivalent to

$$T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \mathbf{0}_W,$$

which implies that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}_V,$$

since T is injective. Since at least one of α_1, α_2, α_3 is not zero, it follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.