1. You are given

$$A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 3 & 2 \\ 3 & 9 & -1 & -1 \\ -4 & -12 & 4 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 11 \\ 3 \\ -2 \end{pmatrix}.$$

a. Is $\mathbf{u}$ a solution of $A\mathbf{x} = \mathbf{b}$?

b. Find then general solution of $A\mathbf{x} = \mathbf{b}$.

2. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

a. Is it true that for every vector $\mathbf{b}$ in $\mathbb{R}^3$ the equation $A\mathbf{x} = \mathbf{b}$ is consistent? Justify your answer.

b. Evaluate AA^T.

c. True or false. (Justify.)

i. A and AA^T have the same column space.

ii. A and AA^T have the same null space.

iii. $\operatorname{Row}(A)$ and $\operatorname{Row}(AA^T)$ are not the same, but they have the same dimension.

3. Find the quadratic polynomial $f(x) = ax^2 + bx + c$ such that $f(-1) = -4$, $f(2) = 5$ and $f(3) = 0$.

4. Given

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 15 \\ 4 \end{pmatrix}.$$

Solve $A\mathbf{x} = \mathbf{b}$ using the given LU factorization.

5. Let

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 1 & 2 \\ 1 & -3 & -3 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}.$$

a. Find A^{-1}.

b. Use your answer to Part a to find U^{-1}.

6. Let A, B and C be 3×3 matrices. It is given that $\det(A) = 3$ and $\det(B) = 7$. It is also given that C is not invertible. For each determinant below, evaluate it or state that there is not enough information to do so.

a. $\det(10A^2B^{-1})$ b. $\det(A + B)$ c. $\det(AC + BC)$

7. Let

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & -4 & 3 \\ 3 & -2 & 1 \end{pmatrix}.$$

a. Find $\det(A)$.

b. Solve $A\mathbf{x} = \mathbf{e}_1$ for x_2 only, using Cramer's Rule.

8. Let

$$\mathscr{S} = \{\, A \in M_{3\times 3} \colon\ \operatorname{rank}(A) \leqslant 2, \}.$$

Answer each of the questions below, and justify your answers.

a. Give an example of a non-zero matrix in $\mathscr{S}$.

b. Does $\mathscr{S}$ contain the zero matrix?

c. Is $\mathscr{S}$ closed under addition?

d. Is $\mathscr{S}$ closed under scalar multiplication?

e. Is $\mathscr{S}$ a subspace of $M_{3\times 3}$?

9. Find a basis of the vector space

$$\mathscr{V} = \{\, \mathbf{p}(t) \in \mathbb{P}_3 \colon \mathbf{p}'(0) = 0 \text{ and } \mathbf{p}''(0) = 0 \,\}.$$

10. Use the words **must**, **might** or **cannot** to complete the statements below, as appropriate.

a. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ spans a plane in $\mathbb{R}^3$, then $\{\mathbf{u}, \mathbf{v}\}$ _________ span the same plane.

b. If the column vectors of a square matrix A span all of $\mathbb{R}^3$, then the determinant of A _________ be zero.

c. The columns of an $n \times n$ elementary matrix _________ be a basis of $\mathbb{R}^n$.

d. If $A \in M_{5\times 6}$, then $\operatorname{rank}(A)$ _________ be a equal to $\operatorname{rank}(A^T)$.

e. For any invertible matrix A, the rank of A _________ be the same as the rank of A^2.

f. If a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is surjective (onto), then A _________ be invertible.

11. Let $\ell = \mathbf{p} + \operatorname{Span}\{\mathbf{u}\}$, where

$$\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

a. Find a matrix A such that $\mathbf{x} \rightsquigarrow A\mathbf{x}$ is a rotation which transforms ℓ into a horizontal line.

b. Find a non-zero matrix B such that $\mathbf{x} \rightsquigarrow B\mathbf{x}$ transforms ℓ into a single point.

12. Let $\ell_1 = \mathbf{p} + \operatorname{Span}\{\mathbf{u}\}$ and $\ell_2 = \mathbf{q} + \operatorname{Span}\{\mathbf{v}\}$, where

$$\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 13 \\ 14 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

a. Find the point of intersection of ℓ_1 and ℓ_2.

b. Find a normal equation of the plane containing ℓ_1 and ℓ_2.

c. Find the distance between ℓ_1 and the origin.

13. Given $A(-2, 1, 0)$, $B(1, 5, 0)$ and $C(4, 0, 1)$.

a. Find the area of the triangle ABC.

b. Find an equation of the line containing A which is perpendicular to the plane containing A, B and C.

c. Find a unit vector which is parallel to $\overrightarrow{AB}$.

d. Find a point between A and B which is two units away from A.

14. Let $\mathscr{P}_1$ be the plane defined by $3x + y - 4z = 10$ and let $\mathscr{P}_2$ be the plane defined by $-x + 3y - z = 5$.

a. Find the cosine of the angle between $\mathscr{P}_1$ and $\mathscr{P}_2$.

b. Find an equation of the line through the origin which is parallel to both $\mathscr{P}_1$ and $\mathscr{P}_2$.

15. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ such that $\|\mathbf{u}\| = 3$, $\|\mathbf{v}\| = 5$ and $\|\mathbf{u} + \mathbf{v}\| = 7$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

16. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Show that if $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \operatorname{proj}_{\mathbf{v}} \mathbf{w}$ then $\mathbf{u} - \mathbf{w}$ is orthogonal to $\mathbf{v}$.

17. Let A be an invertible $n \times n$ matrix and let $T\colon M_{n\times n} \to M_{n\times n}$ be the linear transformation defined by defined by $T(X) = AXA^{-1}$.

a. Show that $\det\big(T(X)\big) = \det(X)$.

b. Show that $T(X)T(Y) = T(XY)$.

1. a. Since $A\mathbf{u} = \mathbf{b}$, $\mathbf{u}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

b. Since

$$A \sim \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(by row reducing or by inspection, since the it is clear from the first three rows that the first, third and fourth columns of A are linearly independent), and since $A\mathbf{u} = \mathbf{b}$, it follows that the general solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{u} + \operatorname{Span}\{-3\mathbf{e}_1 + \mathbf{e}_2\}$.

2. a. Yes, A has three pivot columns so $\operatorname{Col}(A) = \mathbb{R}^3$.

b. The three columns of AA^T are the second column of A, the third column of A and the sum of the fourth and fifth columns of A; *i.e.*, $AA^T = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & 2\mathbf{e}_3 \end{pmatrix}$.

i. True, because $\operatorname{Col}(A)$ and $\operatorname{Col}(AA^T)$ are each equal to $\mathbb{R}^3$.

ii. False, because the null space of A is a (2- dimensional) subspace of $\mathbb{R}^5$ and the null space of AA^T is a subspace (the zero subspace) of $\mathbb{R}^3$.

iii. True, because the row space of A is a 3-dimensional subspace of $\mathbb{R}^5$ and the row space of AA^T is $\mathbb{R}^3$.

3. Since $f(3) = 0$, $f(x) = (x-3)(\alpha x + \beta)$ for some real numbers α and β. Now $f(-1) = -4$ gives $\alpha - \beta = -1$ and $f(2) = 5$ gives $2\alpha + \beta = -5$, so $3\alpha = -6$, or $\alpha = -2$, and $\beta = -1$; therefore, $f(x) = -(x-3)(2x+1) = -2x^2 + 5x + 3$.

4. Solving $L\mathbf{y} = \mathbf{b}$ gives $y_1 = 1$, $y_2 = 15 - 3 = 12$ and $y_3 = 4 + 2 - 6 = 0$. Next, solving $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ gives $x_2 = \frac{1}{4} \cdot 12 = 3$ and $x_1 = \frac{1}{2}(1 + 3) = 2$.

5. a. Reducing A to I_3,

$$A \sim \begin{pmatrix} 2 & 0 & 7 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \sim I_3,$$

and applying the same sequence of elementary row operations to I_3, gives

$$I_3 \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & -2 \end{pmatrix} \sim \begin{pmatrix} -3 & 15 & 7 \\ -2 & 9 & 4 \\ 1 & -4 & -2 \end{pmatrix} = A^{-1}.$$

b. By inspection,

$$U^{-1} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & A^{-1} \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -3 & 15 & 7 \\ 0 & 0 & -2 & 9 & 4 \\ 0 & 0 & 1 & -4 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

6. a. $\det(10A^2B^{-1})10^3 3^2 7^{-1} = \frac{9000}{7}$.

b. There is not enough information to compute $\det(A + B)$.

c. $\det(AC + BC) = \det\big((A + B)C\big) = 0$, since C is singular.

7. Since

$$\det(A) = -5 \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = 15 \quad \text{and} \quad \det A_2(\mathbf{e}_1) = -\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 7,$$

it follows that $x_2 = \frac{7}{15}$.

8. a. The 3×3 matrix $\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0} \end{pmatrix}$ is a non-zero matrix in $\mathscr{S}$, since its rank is 2.

b. The zero matrix has rank $0 \leqslant 2$, so it does belong to $\mathscr{S}$.

c. Let A be the matrix given in Part a and let $B = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{e}_3 \end{pmatrix}$; $\operatorname{rank}(B) = 1 \leqslant 2$ so $A, B \in \mathscr{S}$. However, $A + B = I_3$, whose rank is 3, so $A + B \notin \mathscr{S}$. This shows that $\mathscr{S}$ is not closed under addition.

d. For any matrix X and any scalar α, $\operatorname{Col}(\alpha X)$ is a subspace of $\operatorname{Col}(X)$, which implies that $\operatorname{rank}(\alpha X) \leqslant \operatorname{rank}(X)$. It follows that $\mathscr{S}$ is closed under scalar multiplication.

e. Since $\mathscr{S}$ is not closed under addition, it is not a subspace of $M_{3\times 3}$.

9. If $\mathbf{p}(t)$ is any real polynomial, then $\mathbf{p}'(0) = 0$ and $\mathbf{p}''(0) = 0$ if, and only if, $\mathbf{p}(t) = \mathbf{p}(0) + t^3\mathbf{q}(t)$ for some real polynomial $\mathbf{q}(t)$. It follows that $\{1, t^3\}$ is a basis of the subspace of all such polynomials in $\mathbb{P}_3$.

10. a. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ spans a plane in $\mathbb{R}^3$, then $\{\mathbf{u}, \mathbf{v}\}$ **might** span the same plane.

b. If the column vectors of a square matrix A span all of $\mathbb{R}^3$, then the determinant of A **cannot** be zero.

c. The columns of an $n \times n$ elementary matrix **must** be a basis of $\mathbb{R}^n$.

d. If $A \in M_{5\times 6}$, then $\operatorname{rank}(A)$ **must** be a equal to $\operatorname{rank}(A^T)$.

e. For any invertible matrix A, the rank of A **must** be the same as the rank of A^2.

f. If a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is surjective (onto), then A **must** be invertible.

11. a. If A is the standard matrix of the rotation about the origin through $\frac{1}{4}\pi$, then $A\mathbf{u} = \sqrt{2}\mathbf{e}_1$, so multiplication on the left by $A = \frac{1}{2}\sqrt{2}\begin{pmatrix} \mathbf{e}_1 + \mathbf{e}_2 & -\mathbf{e}_1 + \mathbf{e}_2 \end{pmatrix}$ transforms ℓ into $\sqrt{2}(-\mathbf{e}_1 + 2\mathbf{e}_2) + \operatorname{Span}\{\mathbf{e}_1\}$.

b. Since $(\mathbf{e}_1 + \mathbf{e}_2)^T\mathbf{u} = 0$, multiplying on the left by $B = \begin{pmatrix} \mathbf{e}_1 + \mathbf{e}_2 & \mathbf{e}_1 + \mathbf{e}_2 \end{pmatrix}$ transforms ℓ into $4(\mathbf{e}_1 + \mathbf{e}_2)$.

12. a. A solution of $\mathbf{p} + s\mathbf{u} = \mathbf{q} - t\mathbf{v}$ is obtained by reducing $\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{q} - \mathbf{p} \end{pmatrix}$, *i.e.*,

$$\begin{pmatrix} 2 & 3 & 12 \\ 3 & 2 & 13 \\ -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So lines intersect at

$$\mathbf{p} + 3\mathbf{u} = \mathbf{q} - 2\mathbf{v} = \begin{pmatrix} 7 \\ 10 \\ -2 \end{pmatrix}.$$

b. The vector

$$\mathbf{n} = \tfrac{1}{5}\mathbf{u} \times \mathbf{v} = \tfrac{1}{5}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

is orthogonal to the plane containing ℓ_1 and ℓ_2, and so a normal equation of this plane is $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\mathbf{p}$, *i.e.*, $x - y - z = -1$.

c. The distance between ℓ_1 and the origin is equal to the area of the parallelogram formed by $\mathbf{p}$ and $\mathbf{u}$ divided by the length of $\mathbf{u}$, which is

$$\frac{\|\mathbf{p} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{(-4)^2 + 3^2 + 1^2}}{\sqrt{2^2 + 3^2 + (-1)^2}} = \frac{\sqrt{26}}{\sqrt{14}} = \tfrac{1}{7}\sqrt{91}.$$

13. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of, respectively, A, B and C, and let

$$\mathbf{u} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \ \mathbf{v} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \ \text{and} \ \mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 4 \\ -3 \\ -27 \end{pmatrix}.$$

a. The area of the triangle ABC is equal to

$$\tfrac{1}{2}\|\mathbf{n}\| = \tfrac{1}{2}\sqrt{4^2 + (-3)^2 + 27^2} = \tfrac{1}{2}\sqrt{754}.$$

b. The line which contains A and is perpendicular to the plane containing A, B and C is $\mathbf{a} + \operatorname{Span}\{\mathbf{n}\}$.

c. $\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{5}\mathbf{u}$ is a unit vector parallel to $\overrightarrow{AB}$.

d. The point with position vector

$$\mathbf{a} + \tfrac{2}{5}\mathbf{u} = \tfrac{1}{5}\begin{pmatrix} -4 \\ 13 \\ 0 \end{pmatrix}$$

is between A and B and two units away from A.

14. Let $\mathbf{n}_1$, $\mathbf{n}_2$ be, respectively, the given normal vectors to $\mathscr{P}_1$, $\mathscr{P}_2$.

a. The cosine of the angle between $\mathscr{P}_1$ and $\mathscr{P}_2$ is equal to

$$\frac{|\mathbf{n}_1^T\mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{4}{\sqrt{26}\sqrt{11}} = \tfrac{2}{143}\sqrt{286}.$$

b. The line $\operatorname{Span}\{\mathbf{v}\}$ contains the origin and is parallel to both planes, where

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ 10 \end{pmatrix}.$$

15. Expanding $(\mathbf{u} + \mathbf{v})^T(\mathbf{u} + \mathbf{v})$ gives $49 = 34 + 2\mathbf{u}^T\mathbf{v}$, or $\mathbf{u}^T\mathbf{v} = \frac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\|$, and so the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\arccos\big(\frac{1}{2}\big) = \frac{1}{3}\pi$.

16. $\operatorname{perp}_{\mathbf{v}}(\mathbf{u} - \mathbf{w}) = \mathbf{u} - \mathbf{w} - \big(\operatorname{proj}_{\mathbf{v}}\mathbf{u} - \operatorname{proj}_{\mathbf{v}}\mathbf{w}\big) = \mathbf{u} - \mathbf{w}$, provided $\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \operatorname{proj}_{\mathbf{v}}\mathbf{w}$, which implies that $\mathbf{u} - \mathbf{w}$ is orthogonal to $\mathbf{v}$, as required.

17. a. $\det\big(T(x)\big) = (\det A)(\det X)(\det A)^{-1} = \det(x)$, as required.

b. $T(X)T(Y) = (AXA^{-1})(AYA^{-1}) = AXYA^{-1} = T(XY)$, as required.