1. Let $A = \begin{pmatrix} 1 & 1 & c \\ 1 & c & c \\ c & c & c \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix}$. Find all values of c for which:

a. $A\mathbf{x} = \mathbf{b}$ has a unique solution; b. $A\mathbf{x} = \mathbf{b}$ has no solution;

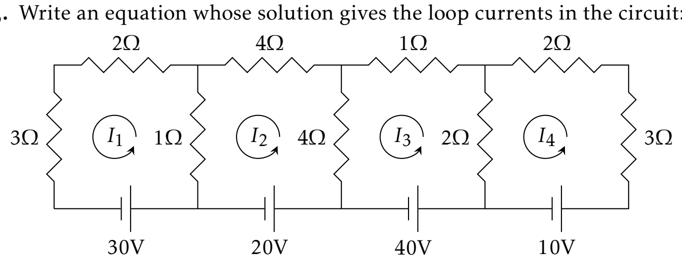
c. $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions;

d. $\mathrm{Col}(A)$ is a plane; e. $\mathrm{Nul}(A)$ is a plane.

2. Find the general solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 5 & 1 & -7 \\ -2 & -10 & 1 & 2 \\ -5 & -25 & 1 & 11 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -3 \\ -9 \end{pmatrix}.$$

3. Write an equation whose solution gives the loop currents in the circuit:



4. Find an LU factorization of the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

5. Let L and A be 3×3 matrices, with L unit lower triangular and $\det(A) = 5$, and let I denote the 3×3 identity matrix. Compute, if possible:

a. $\det\big(2A^T L\big)$ b. $\det\big((A^{-1})^2\big)$ c. $\det(L + A)$ d. $\det(L + 2I)$

6. Solve the following linear system for x_4 only, using Cramer's Rule.

$$\begin{array}{rcrcrcrcl} & - & 2x_2 & + & 2x_3 & - & x_4 & = & 0 \\ -3x_1 & + & 6x_2 & + & 2x_3 & + & x_4 & = & 0 \\ -x_1 & & & - & x_3 & + & x_4 & = & 0 \\ -2x_1 & + & x_2 & & & + & x_4 & = & 1 \end{array}$$

7. Find A^{-1} and express A as a product of elementary matrices, where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ -3 & -3 & 3 \end{pmatrix}.$$

8. Consider the matrix equation $(A - AX)^{-1} = X^{-1}B$.

a. Solve for X symbolically, and note any necessary inversions.

b. Give 3×3 matrices A and B for which the equation has no solution.

9. a. Let M be a matrix such that $M^2 = I$. Prove that $\det(M) = \pm 1$.

b. If N is a square matrix and $\det(N) = 1$, is N^2 necessarily equal to I? Support your answer with a proof or a counterexample.

10. Suppose that A and B are $n \times n$ matrices. Complete each of the following sentences with the word **must**, **might** or **cannot**, as appropriate.

a. If E_1 and E_2 are elementary matrices, then E_1E_2 _____ be equal to E_2E_1.

b. If $A^3 = I$, then A _____ be invertible.

c. If $\det(A) = 0$ is zero then $\mathbf{x} \rightsquigarrow A\mathbf{x} \colon \mathbb{R}^n \to \mathbb{R}^n$ _____ be invertible.

d. The matrix $(I - A)(I + A)$ _____ be equal to $I - A^2$.

e. If B has no column of zeros, but AB does, then the columns of A _____ be linearly independent.

11. Let $\mathscr{S} = \{X \in M_{2\times 2} \colon AX - X = 0\}$, where $A = \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}$.

a. Find a non-zero 2×2 matrices, one in $\mathscr{S}$ and one not in $\mathscr{S}$.

b. Does $\mathscr{S}$ contain the zero 2×2 matrix? Justify your answer.

c. Is $\mathscr{S}$ closed under addition? Justify your answer.

d. Is $\mathscr{S}$ closed under scalar multiplication? Justify your answer.

e. Is $\mathscr{S}$ a subspace of $M_{2\times 2}$?

12. Let $\mathscr{S} = \{s\mathbf{e}_1 + t\mathbf{e}_2 \in \mathbb{R}^2 \colon 0 \leqslant s, t \leqslant 1\}$ and $\mathscr{I} = \{t\mathbf{e}_1 \in \mathbb{R}^2 \colon -1 \leqslant t \leqslant 0\}$, and for $i = 1, 2$ let P_i be the standard matrix of a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ which projects points onto the x_i axis.

a. Write the matrices P_1 and P_2.

b. Find a rotation matrix R_1 such that $\mathbf{x} \rightsquigarrow R_1P_1\mathbf{x}$ transforms $\mathscr{S}$ into $\mathscr{I}$.

c. Do R_1 and P_1 commute?

d. Find a rotation matrix R_2 such that $\mathbf{x} \rightsquigarrow R_2P_2\mathbf{x}$ transforms $\mathscr{S}$ into $\mathscr{I}$.

e. Do R_2 and P_2 commute?

f. Find a basis of the null space of R_2P_2.

13. A matrix A and its reduced echelon form U are given below.

$$A = \begin{pmatrix} 1 & 4 & -2 & 4 & v & 3 & 6 \\ 3 & 12 & -6 & 12 & w & 2 & 15 \\ -2 & -8 & 4 & -8 & x & -1 & -13 \\ 1 & 4 & -2 & 5 & y & 0 & 3 \\ 3 & 12 & -6 & 12 & z & 3 & 10 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 4 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

You may use the following notation in your answers.

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{a}_7 \end{pmatrix} \qquad U^T = \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 \end{pmatrix}$$

a. Is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ a basis of $\mathrm{Col}(A)$? b. Find a basis $\mathscr{C}$ of $\mathrm{Col}(A)$.

c. Find a basis of $\mathrm{Row}(A)$. d. Find a basis of $\mathrm{Nul}(A)$.

e. Express $\mathbf{a}_7$ as a linear combination of the vectors in $\mathscr{C}$ from Part b.

f. Find the values of v, w, x, y and z in the matrix A.

g. What is the dimension of $\mathrm{Nul}(A^T)$?

14. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$, $\mathbf{q}$ be non-zero vectors in $\mathbb{R}^3$, with $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\} = \mathrm{Span}\{\mathbf{p}, \mathbf{q}\}$.

a. Suppose that $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ is a line. Explain why $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$.

b. Suppose that $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ is a plane. Explain why $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$.

15. Let $\mathscr{P}$ be the plane defined by $x_1 + 2x_2 + 2x_3 = 18$, and let $\mathscr{T}$ be the triangle whose vertices are the axis intercepts of $\mathscr{P}$.

a. Find an equation of the line which is perpendicular to $\mathscr{P}$ and contains the origin.

b. Find the point of intersection of the line from Part a with the plane $\mathscr{P}$.

c. Find the distance between the origin and the plane $\mathscr{P}$.

d. Write the vertices of the triangle $\mathscr{T}$.

e. Find the area of the triangle $\mathscr{T}$.

f. Find the distance between $P(2, 2, 2)$ and the line found in Part a.

g. Find the cosine of the (dihedral) angle between the plane $\mathscr{P}$ and the x_1x_2 plane.

16. Let X, Y, Z, W be $n \times n$ matrices, I the $n \times n$ identity matrix,

$$B = \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

a. Compute and simplify B^2. b. Find B^{-1}.

c. Under what conditions on X, Y, Z and W does B commute with C?

17. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be vertices of a triangle centred at the origin in $\mathbb{R}^n$, so that $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$, let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be vertices of a triangle centred at the origin in $\mathbb{R}^m$, so that $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 = \mathbf{0}$, and let $T \colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\mathbf{a}_1) = \mathbf{b}_1$ and $T(\mathbf{a}_2) = \mathbf{b}_2$. Prove that $T(\mathbf{a}_3) = \mathbf{b}_3$.

18. Let $T \colon \mathbb{P}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(p) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$.

a. Find a basis $\mathscr{B}$ of the kernel of T.

b. Compute $T(-6 + 5t + 3t^2 - 2t^3)$.

c. Give the coordinate vector of $-6 + 5t + 3t^2 - 2t^3$ relative to $\mathscr{B}$ (from Part a).

1. First observe that

$$\begin{pmatrix} A & \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & 1 & c & 1 \\ 1 & c & c & 2 \\ c & c & c & c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & c & 1 \\ 0 & c-1 & 0 & 1 \\ 0 & 0 & c-c^2 & 0 \end{pmatrix},$$

so that $\operatorname{rank}(A) = 1$ and $\mathbf{b} \notin \operatorname{Col}(A)$ if $c = 1$, $\operatorname{rank}(A) = 2$ and $\mathbf{b} \in \operatorname{Col}(A)$ if $c = 0$, and otherwise $\operatorname{rank}(A) = 3$, so $\operatorname{Col}(A) = \mathbb{R}^3$.

a. $A\mathbf{x} = \mathbf{b}$ has a unique solution if, and only if, $\operatorname{rank}(A) = 3$; *i.e.*, $c \neq 0, 1$.

b. $A\mathbf{x} = \mathbf{b}$ has no solution if, and only if, $\mathbf{b} \notin \operatorname{Col}(A)$; *i.e.*, $c = 1$.

c. $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if, and only if, $\operatorname{rank}(A) < 3$ and $\mathbf{b} \in \operatorname{Col}(A)$; *i.e.*, $c = 0$.

d. $\operatorname{Col}(A)$ is a plane if, and only if, $\operatorname{rank}(A) = 2$; *i.e.*, $c = 0$.

e. $\operatorname{Nul}(A)$ is a plane if, and only if, $\operatorname{rank}(A) = 1$; *i.e.*, $c = 1$.

2. Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{pmatrix}$; by inspection $\mathbf{a}_1$, $\mathbf{a}_3$ are linearly independent, $\mathbf{a}_2 = 5\mathbf{a}_1$, $\mathbf{a}_4 = -3\mathbf{a}_1 - 4\mathbf{a}_3$ and $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_3$. So the general solution of $A\mathbf{x} = \mathbf{b}$ is equal to $\mathbf{p} + \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{p} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 1 \end{pmatrix}.$$

3. By Kirchhoff's and Ohm's Laws, the currents in the circuit are the entries of the solution $\mathbf{I}$ of $R\mathbf{I} = \mathbf{V}$, where

$$R = \begin{pmatrix} 6 & -1 & 0 & 0 \\ -1 & 9 & -4 & 0 \\ 0 & -4 & 7 & -2 \\ 0 & 0 & -2 & 7 \end{pmatrix} \qquad \text{and} \qquad \mathbf{V} = \begin{pmatrix} 30 \\ 20 \\ 40 \\ 10 \end{pmatrix}.$$

4. An LU factorization of the given matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

via the rough work

$$\boxed{\begin{array}{cc} -1 & -1 \\ -2 & 0 \\ 1 & -1 \end{array}} \quad \rightsquigarrow \quad \boxed{\begin{array}{c} 2 \\ -2 \end{array}}.$$

5. Recall that the determinant is multilinear, preserves products, is invariant under transposition, and that the determinant of a square triangular matrix is the product of its diagonal entries.

a. $\det\left(2A^T L\right) = 2^3 \det(A)\det(L) = 2^3(5)(1) = 40$

b. $\det\left((A^{-1})^2\right) = \left(\det(A)\right)^{-2} = 5^{-2} = \frac{1}{25}$

c. $\det(L + A)$ is not determined by the given information. For example, if $A = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & 5\mathbf{e}_3 \end{pmatrix}$ then $\det(L + A) = 2 \cdot 2 \cdot 6 = 24$. On the other hand, if $A = \begin{pmatrix} -\mathbf{e}_1 & -\mathbf{e}_2 & 5\mathbf{e}_3 \end{pmatrix}$ then $\det(L + A) = 0$.

d. $\det(L + 2I) = 3 \cdot 3 \cdot 3 = 27$

6. Since

$$\begin{vmatrix} 0 & -2 & 2 & -1 \\ -3 & 6 & 2 & 1 \\ -1 & 0 & -1 & 1 \\ -2 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & -1 & -1 \\ -2 & 6 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & -2 & 1 \\ -2 & 6 & 3 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & -3 & 0 \\ -2 & 4 & 1 \\ -1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} = -3,$$

and

$$\begin{vmatrix} 0 & -2 & 2 & 0 \\ -3 & 6 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 2 \\ -3 & 6 & 2 \\ -1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ -3 & 8 & 2 \\ -1 & -1 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} -3 & 8 \\ -1 & -1 \end{vmatrix} = 22,$$

it follows that $x_4 = -\frac{22}{3}$.

7. The inverse of A is obtained by row reducing A to I_3,

$$A \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim I_3,$$

and applying the same elementary row operations to I_3,

$$I_3 \sim \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} -3 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \\ -2 & 1 & 0 \end{pmatrix} = A^{-1};$$

A is the product of the inverses of the corresponding elementary matrices, taken in the order of application of the elementary row operations:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. The given equation is equivalent to $A - AX = B^{-1}X$, or $(A + B^{-1})X = A$, which gives $X = (A + B^{-1})^{-1}A$, provided A, B and $A + B^{-1}$ are invertible. If $A = I_3$ and $B = -I_3$, then the equation in question is $(I_3 - X)^{-1} = -X^{-1}$, or equivalently $I_3 - X = -X$, which plainly has no solution.

9. a. If $M^2 = I$, then $1 = \det(I) = \det(M^2) = (\det M)^2$, so $\det(M) = \pm 1$.

b. If $N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $\det(N) = 1$, but $N^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \neq I_2$.

10. a. If E_1, E_2 are elementary matrices, then E_1E_2 **might** be equal to E_2E_1.

b. If $A^3 = I$, then A **must** be invertible.

c. If $\det(A)$ is zero, then the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ **cannot** be invertible.

d. The matrix $(I - A)(I + A)$ **must** be equal to $I - A^2$.

e. If B has no column of zeros, but AB does, then the columns of A **cannot** be linearly independent.

11. Let $T\colon M_{2\times 2} \to M_{2\times 2}$ be the linear transformation defined by

$$T(X) = BX, \qquad \text{where} \qquad B = A - I_2 = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix};$$

then $\mathscr{S}$ is the kernel of T, and a 2×2 matrix belongs to $\mathscr{S}$ if, and only if, each of its columns belongs to the null space of B, *i.e.*,

$$\mathscr{S} = \operatorname{Span}\left\{ \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \right\}.$$

Either of the spanning vectors (or any non-trivial linear combination of them) is an example of a non-zero matrix which belongs to $\mathscr{S}$, and the 2×2 identity matrix is a non-zero 2×2 matrix which does not belong to $\mathscr{S}$.

As $\mathscr{S}$ is the kernel of a linear transformation, it is a subspace of $M_{2\times 2}$, so the answers to Parts c through f are all yes.

12. a. $P_1 = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}$, where a is a real number.

b. If $a = 0$ then, since $P_1(\mathscr{S}) = -\mathscr{I}$, $R_1 = -I_2$ is the required rotation matrix. If $a \neq 0$, there is no such rotation matrix.

c. If $a = 0$ then, since every 2×2 matrix commutes with $-I_2$, R_1 and P_1 do commute. If $a \neq 0$ then there is no R_1 and thus no question.

d. If $a = 0$ then, as $P_2(\mathscr{S}) = \{t\mathbf{e}_2 : 0 \leqslant t \leqslant 1\}$, $R_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the required rotation matrix. If $a \neq 0$ then there is no such rotation matrix.

e. If $a = 0$ then, as $P_2R_2\mathbf{e}_1 = \mathbf{e}_2 \neq \mathbf{0} = R_2P_2\mathbf{e}_1$, R_2 and P_2 do not commute. If $a \neq 0$ then there is no R_2 and thus no question.

f. If $a = 0$ then, since R_2 is invertible, $\{\mathbf{e}_1\}$ is a basis of $\mathrm{Nul}(R_2P_2) = \mathrm{Nul}(P_2)$. If $a \neq 0$ then there is no R_2 and thus no null space of R_2P_2.

13. a. Since $\mathbf{a}_2 = 4\mathbf{a}_1$, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is linearly dependent, so it is not a basis of $\mathrm{Col}(A)$.

b. The list $\{\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$ (of pivot columns of A) is a basis of $\mathrm{Col}(A)$.

c. The list $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ (of non-zero columns of U^T) is a basis of $\mathrm{Row}(A)$.

d. From the answer to Part b and $\mathbf{a}_2 = 4\mathbf{a}_1$, $\mathbf{a}_3 = -2\mathbf{a}_1$ and $\mathbf{a}_7 = \mathbf{a}_1 - 2\mathbf{a}_5 + 3\mathbf{a}_6$, it follows that

$$\left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 2 \\ -3 \\ 1 \end{pmatrix} \right\}$$

is a basis of the null space of A.

e. As above, $\mathbf{a}_7 = \mathbf{a}_1 - 2\mathbf{a}_5 + 3\mathbf{a}_6$.

f. From the previous part of this question, $\mathbf{a}_5 = \frac{1}{2}(\mathbf{a}_1 + 3\mathbf{a}_6 - \mathbf{a}_7)$, so $v = 2$, $w = -3$, $x = 4$, $y = -1$ and $z = 1$.

g. By the rank formula, $\dim\big(\mathrm{Nul}(A^T)\big) = 5 - \mathrm{rank}(A^T) = 5 - 4 = 1$.

14. a. If $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ is a line then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, and hence $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$.

b. If $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\} = \mathrm{Span}\{\mathbf{p}, \mathbf{q}\}$ is a plane, then the normal vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{p} \times \mathbf{q}$ are (non-zero and) parallel; thus $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{p} \times \mathbf{q}) = \mathbf{0}$.

15. a. The line $\mathrm{Span}\{\mathbf{n}\}$ is perpendicular to $\mathscr{P}$ and contains $\mathbf{0}$, where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

b. Since $\mathbf{n}^T\mathbf{n} = 9$, the line from Part a meets the plane $\mathscr{P}$ at $2\mathbf{n}$.

c. The distance between the origin and the plane $\mathscr{P}$ is $\|2\mathbf{n}\| = 6$.

d. The vertices of $\mathscr{T}$ are $18\mathbf{e}_1$, $9\mathbf{e}_2$ and $9\mathbf{e}_3$ (the axis intercepts of $\mathscr{P}$).

e. The area of $\mathscr{T}$ is $\frac{81}{2}\|(2\mathbf{e}_1 - \mathbf{e}_3) \times (\mathbf{e}_2 - \mathbf{e}_3)\| = \frac{81}{2}\|-\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3\| = \frac{243}{2}$.

f. The distance between $P(2,2,2)$ and the line from Part a is equal to the length of

$$\mathbf{p} \times \mathbf{n} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

divided by $\|\mathbf{n}\| = 3$, or $\frac{2}{3}\sqrt{2}$.

g. The cosine of the dihedral angle between $\mathscr{P}$ and the x_1x_2 plane is

$$\frac{|\mathbf{n}^T\mathbf{e}_3|}{\|\mathbf{n}\|\|\mathbf{e}_3\|} = \frac{2}{3}.$$

16. a. $B^2 = \begin{pmatrix} I & -I \\ I & I \end{pmatrix}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} I^2 - I^2 & -I^2 - I^2 \\ I^2 + I^2 & -I^2 + I^2 \end{pmatrix} = 2\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

b. The result of Part a implies that $B^{-1} = \frac{1}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$.

c. Since

$$BC = \begin{pmatrix} I & -I \\ I & I \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} X - Z & Y - W \\ X + Z & Y + W \end{pmatrix}$$

and

$$CB = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} X + Y & -X + Y \\ Z + W & -Z + W \end{pmatrix},$$

it follows that B commutes with C if, and only if, $Z = -Y$ and $W = X$.

17. Since $\mathbf{a}_3 = -\mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{b}_3 = -\mathbf{b}_1 - \mathbf{b}_2$, the linearity of T gives

$$T(\mathbf{a}_3) = T(-\mathbf{a}_1 - \mathbf{a}_2) = -T(\mathbf{a}_1) - T(\mathbf{a}_2) = -\mathbf{b}_1 - \mathbf{b}_2 = \mathbf{b}_3.$$

18. a. A polynomial $p(t)$ satisfies $p(1) = p(2) = 0$ if, and only if, $(t-1)(t-2)$ is a factor of $p(t)$. So $\mathscr{B} = \{(t-1)(t-2), t(t-1)(t-2)\}$ is a basis of $\ker(T)$.

b. A direct calculation gives $T(-6 + 5t + 3t^2 - 2t^3) = \begin{pmatrix} -6 + 5 + 3 - 2 \\ -6 + 10 + 12 - 16 \end{pmatrix} = \mathbf{0}$.

c. Inspecting the constant and cubic coefficients gives

$$\big[-6 + 5t + 3t^2 - 2t^3\big]_{\mathscr{B}} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}.$$

This result could also be obtained by factorizing

$$-6 + 5t + 3t^2 - 2t^3 = (t-1)(t-2)(-3-2t).$$