1. You are given $A = \begin{pmatrix} 1 & 1 & 3 & 3 \\ -1 & -1 & -3 & -3 \\ -2 & -1 & -4 & -3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ 6 \\ 7 \\ -5 \end{pmatrix}$.

a. Find the general solution of $A\mathbf{x} = \mathbf{b}$.

b. Find the specific solution for which $x_1 = x_2$ and $x_3 = x_4$.

c. Which columns of A, if any, are in the solution set of $A\mathbf{x} = \mathbf{b}$.

d. Which columns of A, if any, are in the null space of A.

e. Find a basis for the row space of A.

2. Let A and B be 4×4 matrices with $\det(A) = 3$ and $\det(B) = -2$. If possible, compute:

a. $\det\big((2A)^{-1}\big)$ b. $\det(B^{-1}A^TB)$ c. $\det(B + B^{-1})$

3. Let $A = \begin{pmatrix} 2 & -2 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 6 & 2 \end{pmatrix}$.

a. Find $\det(A)$

b. How many solutions does $A\mathbf{x} = \mathbf{0}$ have?

4. Show that $\det(A) = 0$ if A is a skew-symmetic $n \times n$ matrix with n odd.

5. The graph of a quadratic polynomial contains the point $(2,-1)$ and is tangent to the line defined by $2x - y = 8$ where $x = 1$. Write a matrix equation whose solution gives the coefficients of the polynomial, and use Cramer's rule to find the constant term only.

6. Given that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$, determine all values of k so that the vectors $\mathbf{u}+2\mathbf{v}$, $\mathbf{v}+3\mathbf{w}$, $k\mathbf{u}+\mathbf{w}$ are linearly independent.

7. a. Given nonsingular matrices B and D, find the inverse of $\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$.

b. Use Part a to give the inverse of $\begin{pmatrix} -3 & 2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & -2 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}$.

8. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 9 & 1 \\ 6 & -8 & k \end{pmatrix}$.

a. Find an LU decomposition of A.

b. Compute the determinant of A.

c. For which k is A not invertible?

d. Express L as a product of elementary matrices.

9. Let V be the set of all 2×2 upper triangular matrices.

a. What is the dimension of V?

b. Express $\begin{pmatrix} 19 & 20 \\ 0 & -3 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$.

c. Does $\left\{ \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 19 & 20 \\ 0 & -3 \end{pmatrix} \right\}$ span V? Justify your answer.

10. In $\mathbb{R}^3$ let $\mathbf{u} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 \in \mathbb{R}^3$, and let $W = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{u}^T\mathbf{x} = 0\}$. Given that W is a subspace of $\mathbb{R}^3$, find a basis for W.

11. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear tranformation which satisfies

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Find the standard matrix of T. Find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$.

12. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - 2b \\ b^2 - a \end{pmatrix}$.

a. Evaluate $T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $T\begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

b. Explain why the results in part a imply that T is not linear.

c. Find a non-zero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

13. Given the points $A(5,2,0)$, $B(7,0,-2)$, $C(2,1,1)$ and $D(4,3,4)$.

a. Find a vector of length 2 which is parallel to the vector $\overrightarrow{AB}$.

b. Find an equation of the line AB.

c. Find the distance between D and the line AB.

d. Find the point on the line AB which is closest to D.

e. Find the area of the triangle ABC.

14. Let V be the set of 2×2 singular matrices.

a. Is V closed under scalar multiplication? Justify.

b. Is V closed under addition? Justify.

15. Let A be a 2×2 reflection matrix and let B be a 2×2 rotation matrix. Complete each of the following sentences with **must**, **might** or **cannot**.

a. A^2 ________ equal A. b. A^{-1} ________ equal A.

c. B^3 ________ equal B. d. $\det(A^2)$ ________ equal $\det(B^2)$.

16. Let $\ell_1 = \mathbf{p} + \mathrm{Span}\{\mathbf{u}\}$ and $\ell_2 = \mathbf{q} + \mathrm{Span}\{\mathbf{v}\}$, where

$$\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

a. Find the point of intersection of ℓ_1 and ℓ_2.

b. Find the cosine of the acute angle formed by the lines.

c. Find an implicit equation of the plane containing ℓ_1 and ℓ_2.

d. Find the x-intercept of the plane from part c.

17. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ such that $\mathbf{u} + 2\mathbf{v}$ is orthogonal to $\mathbf{u} - 2\mathbf{v}$, and $\|\mathbf{u}\| = 1$. Find $\|\mathbf{v}\|$.

18. Give an equation of the line which contains the point $P(1,5,2)$ and is parallel to the planes defined by $x_1 + 2x_2 + x_3 = 4$ and $2x_1 + 5x_2 + 3x_3 = 1$.

1. a. Write $A = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4\end{pmatrix}$. By inspection, the first two columns of A are linearly independent, $\mathbf{a}_3 = \mathbf{a}_1 + 2\mathbf{a}_2$, $\mathbf{a}_4 = 3\mathbf{a}_2$ and $\mathbf{b} = -\mathbf{a}_1 - 5\mathbf{a}_2$, so the solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{p} + \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{p} = \begin{pmatrix}-1\\-5\\0\\0\end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix}-1\\-2\\1\\0\end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix}0\\-3\\0\\1\end{pmatrix}.$$

b. The solution $\mathbf{p} - \mathbf{u} - \mathbf{v} = -\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ is as required.

c. The columns $\mathbf{a}_1 = \mathbf{p} - 2\mathbf{u}$ and $\mathbf{a}_3 = \mathbf{p} - 4\mathbf{u} + 2\mathbf{v}$ are in the solution of $A\mathbf{x} = \mathbf{b}$.

d. The columns $\mathbf{a}_2 = -\mathbf{u} + \mathbf{v}$ and $\mathbf{a}_4 = 3\mathbf{a}_2$ are in the solution of $A\mathbf{x} = \mathbf{0}$.

e. Writing $A^T = \begin{pmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4\end{pmatrix}$, the vectors $\mathbf{r}_1$, $\mathbf{r}_4$ form a basis of $\mathrm{Row}(A)$, since they are plainly linearly independent and $\mathrm{rank}(A) = 2$.

2. a. By multilinearity and product preservation, $\det\big((2A)^{-1}\big) = \frac{1}{2^4 \cdot 3} = \frac{1}{48}$.

b. Product preservation and invariance under transposition implies that $\det(B^{-1}A^TB) = \det(A) = 3$, by

c. The determinant of $B + B^{-1}$ is undetermined. For example, if $B = I$ then $\det(B + B^{-1}) = 16$, and if $B = \begin{pmatrix}-\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\end{pmatrix}$ then $\det(B + B^{-1}) = -16$.

3. A direct calculation gives

$$\det(A) = \begin{vmatrix}0 & 0 & 2 & 1\\0 & 2 & 1 & 1\\1 & -1 & 0 & 1\\0 & 0 & 6 & 2\end{vmatrix} = \begin{vmatrix}0 & 2 & 1\\2 & 1 & 1\\0 & 6 & 2\end{vmatrix} = -2\begin{vmatrix}2 & 1\\6 & 2\end{vmatrix} = 4.$$

Since A is nonsingular, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution (*i.e.*, $\mathbf{0}$).

4. If A is a skew-symmetric $n \times n$ matrix and n is odd, then

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A),$$

so $\det(A) = 0$.

5. If the graph of $p(x) = a + bx + cx^2$ contains $(2, -1)$ and is tangent to the line defined by $2x - y = 8$ where $x = 1$, then $p(2) = -1$, $p(1) = -6$ and $p'(1) = 2$, which is equivalent to

$$\begin{pmatrix}1 & 2 & 4\\1 & 1 & 1\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-1\\-6\\2\end{pmatrix}; \quad \text{also,} \quad a = \frac{\begin{vmatrix}-1 & 2 & 4\\-6 & 1 & 1\\2 & 1 & 2\end{vmatrix}}{\begin{vmatrix}1 & 2 & 4\\1 & 1 & 1\\0 & 1 & 2\end{vmatrix}} = -\frac{\begin{vmatrix}-11 & -23\\5 & 10\end{vmatrix}}{\begin{vmatrix}-1 & -3\\1 & 2\end{vmatrix}} = -5$$

by Cramer's rule.

6. Since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent and

$$\begin{pmatrix}\mathbf{u} + 2\mathbf{v} & \mathbf{v} + 3\mathbf{w} & k\mathbf{u} + \mathbf{w}\end{pmatrix} = \begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix}\begin{pmatrix}1 & 0 & k\\2 & 1 & 0\\0 & 3 & 1\end{pmatrix},$$

the vectors $\mathbf{u} + 2\mathbf{v}$, $\mathbf{v} + 3\mathbf{w}$, $k\mathbf{u} + \mathbf{w}$ are linearly independent if, and only if,

$$\begin{vmatrix}1 & 0 & k\\2 & 1 & 0\\0 & 3 & 1\end{vmatrix} = \begin{vmatrix}1 & -2k\\3 & 1\end{vmatrix} = 1 + 6k \neq 0; \qquad \text{equivalently,} \qquad k \neq -\tfrac{1}{6}.$$

7. For nonsingular B and D, the equation

$$\begin{pmatrix}I & 0\\0 & I\end{pmatrix} = \begin{pmatrix}B & 0\\C & D\end{pmatrix}\begin{pmatrix}U & X\\V & Y\end{pmatrix} = \begin{pmatrix}BU & BX\\CU + DV & CX + DY\end{pmatrix}$$

is equivalent to $U = B^{-1}$, $X = 0$, $CB^{-1} + DV = 0$ and $Y = D^{-1}$. For square matrices M and N, $MN = I$ implies that $NM = I$, so it follows that

$$\begin{pmatrix}B & 0\\C & D\end{pmatrix}^{-1} = \begin{pmatrix}B^{-1} & 0\\-D^{-1}CB^{-1} & D^{-1}\end{pmatrix}.$$

In the case at hand, the blocks of the inverse are

$$B^{-1} = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}, \qquad D^{-1} = \begin{pmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

(each obtained by inspection), and

$$-D^{-1}CB^{-1} = -\begin{pmatrix}3 & 6\\1 & 2\\1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 3\end{pmatrix} = -\begin{pmatrix}15 & 24\\5 & 8\\5 & 8\end{pmatrix}.$$

(The top right block is the zero 2×3 matrix.)

8. An LU factorizton of A is

$$A = \begin{pmatrix}1 & 0 & 0\\2 & 1 & 0\\6 & -4 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & -1\\0 & 5 & 3\\0 & 0 & k + 18\end{pmatrix}, \qquad \text{via} \qquad \begin{matrix}5 & 3\\-20 & k+6\end{matrix} \rightsquigarrow k + 18.$$

The derminant of A is equal to the determinant of U, or $5(k + 18)$, so that A is singular (not invertible) if, and only if, $k = -18$. Also,

$$\begin{pmatrix}1 & 0 & 0\\2 & 1 & 0\\6 & -4 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\6 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -4 & 1\end{pmatrix}$$

is an elementary factorization of L.

9. The matrices

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

form a basis of the linear space V of 2×2 upper triangular matrices, so the dimension of V is 3. Since $7 \cdot 2 - 5(-1) = 19$, $7 \cdot 5 - 5 \cdot 3 = 10$ and $7 \cdot 1 - 5(-2) = -3$, it follows that

$$7\begin{pmatrix}2 & 5\\0 & 1\end{pmatrix} - 5\begin{pmatrix}-1 & 3\\0 & 2\end{pmatrix} = \begin{pmatrix}19 & 20\\0 & -3\end{pmatrix}.$$

Thus, the matrices

$$\begin{pmatrix}2 & 5\\0 & 1\end{pmatrix}, \qquad \begin{pmatrix}-1 & 3\\0 & 2\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}19 & 20\\0 & -3\end{pmatrix}$$

span a 2 dimensional subspace of V; they do not span V.

10. If

$$\mathbf{u} = \begin{pmatrix}1\\2\\2\end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix}2\\-1\\0\end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix}2\\0\\-1\end{pmatrix},$$

then W is the set of all vectors in $\mathbb{R}^3$ which are orthogonal to $\mathbf{u}$, and $\{\mathbf{v}, \mathbf{w}\}$ is a basis of W.

11. The standard matrix A of T satisfies

$$A\begin{pmatrix}1 & 4\\1 & 6\end{pmatrix} = \begin{pmatrix}2 & 5\\1 & 3\end{pmatrix},$$

or equivalently (inverting the right factor on the left)

$$A = \tfrac{1}{2}\begin{pmatrix}2 & 5\\1 & 3\end{pmatrix}\begin{pmatrix}6 & -4\\-1 & 1\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}7 & -3\\3 & -1\end{pmatrix}.$$

Then (by inverting A)

$$T(\mathbf{u}) = \begin{pmatrix}-2\\10\end{pmatrix} \quad \text{if, and only if,} \quad \mathbf{u} = \begin{pmatrix}-1 & 3\\-3 & 7\end{pmatrix}\begin{pmatrix}-2\\10\end{pmatrix} = \begin{pmatrix}32\\76\end{pmatrix}.$$

12. Direct calculations using the definition of T give

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-3\\3\end{pmatrix} \qquad \text{and} \qquad T\begin{pmatrix}3\\6\end{pmatrix} = \begin{pmatrix}-9\\33\end{pmatrix} \neq 3T\begin{pmatrix}1\\2\end{pmatrix},$$

so T does not preserve scalar multiplication. Also, the calculation

$$T\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

is a gives a non-zero vector $\mathbf{x}$ as required. (This vector obtained by solving $0 = a - 2b = a - b^2$, which gives $b = 0$ or $b = 2$.)

13. Let

$$\mathbf{u} = \tfrac{1}{2}\overrightarrow{AB} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}, \qquad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix}-3\\-1\\1\end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \overrightarrow{AD} = \begin{pmatrix}-1\\1\\4\end{pmatrix}.$$

a. Since $\|\mathbf{u}\| = \sqrt{3}$, the vectors

$$\pm\hat{\mathbf{u}} = \pm\tfrac{2}{3}\sqrt{3}\mathbf{u} = \pm\tfrac{2}{3}\sqrt{3}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

have length 2 and are parallel to $\overrightarrow{AB}$.

b. The line AB is equal to $\overrightarrow{OA} + \text{Span}\left\{\tfrac{1}{2}\mathbf{u}\right\} = \begin{pmatrix}5\\2\\0\end{pmatrix} + \text{Span}\left\{\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\}$.

c. The distance between D and the line AB is the length of

$$\text{perp}_{\mathbf{u}}\,\mathbf{w} = \mathbf{w} - \text{proj}_{\mathbf{u}}\,\mathbf{w} = \mathbf{w} - \frac{\mathbf{u}^T\mathbf{w}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} = \begin{pmatrix}-1\\1\\4\end{pmatrix} + 2\begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix},$$

which is $\sqrt{6}$.

d. The point on the line AB which is closest to D is

$$\overrightarrow{OA} + \text{proj}_{\mathbf{u}}\,\mathbf{w} = \begin{pmatrix}5\\2\\0\end{pmatrix} - 2\begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}3\\4\\2\end{pmatrix}.$$

e. The area of triangle ABC is the length of

$$\tfrac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix}1\\-1\\-1\end{pmatrix} \times \begin{pmatrix}-3\\-1\\1\end{pmatrix} = \begin{pmatrix}-2\\2\\-4\end{pmatrix},$$

which is $2\sqrt{6}$.

14. If $A \in V$ and $\alpha \in \mathbb{R}$ then $\det(A) = 0$, and thus $\det(\alpha A) = \alpha^2\det(A) = 0$, so $\alpha A \in V$. So V is closed under scalar multiplication. On the other hand,

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \in V, \qquad \text{but} \qquad \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = I_2 \notin V.$$

So V is not closed under addition.

15. Here A is a 2×2 reflection matrix and B be a 2×2 rotation matrix.

a. A^2 **cannot** equal A.
b. A^{-1} **must** equal A.
c. B^3 **might** equal B.
d. $\det(A^2)$ **must** equal $\det(B^2)$

16. a. The equation $\mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$ is solved for $s, -t$ by reducing

$$\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{q}-\mathbf{p}\end{pmatrix} \sim \begin{pmatrix}1&2&-2\\0&3&-6\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&2\\0&1&-2\\0&0&0\end{pmatrix},$$

which gives the point of intersection

$$\mathbf{p} + 2\mathbf{u} = \mathbf{q} + 2\mathbf{v} = \begin{pmatrix}4\\-1\\6\end{pmatrix}.$$

b. The cosine of the acute angle formed by ℓ_1 and ℓ_2 is

$$\frac{|\mathbf{u}^T\mathbf{v}|}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{7}{(\sqrt{11})(3)} = \tfrac{7}{33}\sqrt{11}.$$

c. The vector

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix}1\\-1\\3\end{pmatrix} \times \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-5\\4\\3\end{pmatrix}$$

is a normal vector to the plane containing ℓ_1 and ℓ_2, so this plane is defined by $5x - 4x - 3z = 6$ (the right side is obtained by evaluating at $\mathbf{p}$, or $\mathbf{q}$).

d. The x intecept is the point on the plane where $y = z = 0$, which is $\tfrac{6}{5}\mathbf{e}_1$.

17. If $\mathbf{u} + 2\mathbf{v}$ is orthogonal to $\mathbf{u} - 2\mathbf{v}$ and $\|\mathbf{u}\| = 1$, then

$$0 = (\mathbf{u} + 2\mathbf{v})^T(\mathbf{u} - 2\mathbf{v}) = \mathbf{u}^T\mathbf{u} - 4\mathbf{v}^T\mathbf{v} = 1 - 4\|\mathbf{v}\|^2,$$

which implies that $\|\mathbf{v}\| = \tfrac{1}{2}$.

18. The vector

$$\begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}2\\5\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix},$$

since it is perpendicular to the normals of the given planes, is parallel to both of them. Therefore, the line which contains $P(1,5,2)$ and is parallel to both planes is

$$\begin{pmatrix}1\\5\\2\end{pmatrix} + \text{Span}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}.$$