1. Given

$$A = \begin{pmatrix} 1 & 1 & 4 & 1 & 6 \\ 2 & 2 & 5 & -1 & 18 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}.$$

a. Solve $A\mathbf{x} = \mathbf{b}$. Express the solution in parametric vector form.

b. If $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{d}$, then give the general solution of $A\mathbf{x} = \mathbf{d}$.

2. Balance the chemical equation $KClO_3 \to KCl + O_2$.

3. Let

$$A = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 3 & 1 \\ -2 & -3 & 0 \end{pmatrix}.$$

a. Find the inverse of A.

b. What is $(A^T)^{-1}$?

4. Compute $\det\begin{pmatrix} 2a+3b & abd - b^2c \\ 2c+3d & ad^2 - bcd \end{pmatrix}$, given that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 8$.

5. You are given the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 9 & 2 & 5 \\ -6 & 22 & 10 \end{pmatrix}.$$

a. Write an LU factorization of A.

b. Write the matrix L as a product of elementary matrices.

6. In this question, you are given that $A \sim R$, where

$$A = \begin{pmatrix} 1 & a & 2 & 1 & e \\ 2 & b & 3 & 2 & f \\ 3 & c & -1 & -1 & g \\ -4 & d & 4 & 1 & h \end{pmatrix}, \; R = \begin{pmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathbf{u} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \; \mathbf{v} = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}, \; \mathbf{w} = \begin{pmatrix} -25 \\ 13 \\ 5 \\ k \end{pmatrix}.$$

a. What are the vectors $\mathbf{u}$ and $\mathbf{v}$?

b. Find a basis of $\mathrm{Col}(A)$.

c. How many vectors are in $\mathrm{Col}(A)$?

d. Find a basis of $\mathrm{Nul}(A)$.

e. For which k, if any, is $\mathbf{w} \in \mathrm{Nul}(A^T)$?

f. Is $\mathrm{Nul}(A^T)$ a line? Justify.

7. Given that A, B, C and X are $n \times n$ matrices, and that $X + A$ and C are invertible, solve the equation $B(X+A)^{-1} = C$ for X.

8. You are given that $A = LU$, L is a 4×4 unit lower triangular matrix, U is a 4×4 upper triangular matrix and $\det(A) = -3$. Evaluate each of the following.

a. $\det(L)$ b. $\det(U)$ c. $\det\left(2(A^T)^3 A^{-1}\right)$ d. $\det(LA + A)$

9. Suppose that A is an $n \times n$ matrix. Show that if $\mathrm{Nul}(A)$ has dimension zero, then $\mathrm{Nul}(A^2)$ must also have dimension zero.

10. Find a non-invertible 2×2 matrix A, such that $\det(A + I) = 0$.

11. Let $\mathscr{H} = \{A \in M_{2\times2} \colon AX = 0\}$, where $X = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$.

a. Find a specific non-zero matrix in $\mathscr{H}$.

b. Given that $\mathscr{H}$ is a subspace of $M_{2\times2}$, give a basis of $\mathscr{H}$.

12. Let $\mathscr{V} = \mathrm{Span}\{I, M\}$ and $\mathscr{W} = \mathrm{Span}\{I, M, N\}$, where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

a. Show that if $A \in \mathscr{V}$ then $A^2 \in \mathscr{W}$.

b. True or false: $\mathscr{V}$ is a two dimensional subspace of $\mathscr{W}$.

c. True or false: $\mathscr{W}$ is a three dimensional subspace of $\mathscr{V}$.

13. Let

$$\mathscr{V} = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} : w = z \text{ and } xy = z^2 \right\}.$$

a. Does the zero vector of $\mathbb{R}^4$ belong to $\mathscr{V}$?

b. Find a non-zero vector in $\mathscr{V}$.

c. Is $\mathscr{V}$ closed under scalar multiplication? Justify your answer.

d. Is $\mathscr{V}$ closed under addition? Justify your answer.

e. Is $\mathscr{V}$ a subspace of $\mathbb{R}^4$?

14. Let $\mathscr{T}$ be the triangle with vertices $A(-2,6,8)$, $B(-3,9,12)$ and $C(0,6,9)$.

a. Is the internal angle of $\mathscr{T}$ at B acute, or obtuse, or a right angle?

b. Find a cartesian equation of the plane which contains the point $P(1,0,-1)$ and is parallel to the plane containing $\mathscr{T}$.

15. Find the distance between $\begin{pmatrix} 7 \\ 10 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}\right\}$.

16. Consider the line ℓ in $\mathbb{R}^3$ given by

$$\mathbf{x} = \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

and the plane $\mathscr{P}$ defined by $x - 2y + 2z = -8$.

a. Find the points on ℓ which are one unit from $\mathscr{P}$.

b. Find the point where ℓ and $\mathscr{P}$ intersect.

17. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates vectors clockwise around the origin by ϑ, then reflects through the x-axis, then rotates again by ϑ clockwise, and finally reflects through the y-axis. Find the standard matrix of T.

18. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -27 \\ 13 \end{pmatrix} \qquad \text{and} \qquad T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -27 \\ 13 \end{pmatrix}.$$

a. Is T injective? Explain your answer.

b. Express $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

c. Is the transformation T linear? Justify your answer.

19. Let $T \colon U \to V$ be a linear transformation. Show that if $T(\mathbf{u}_1) = T(\mathbf{u}_2)$ then $2\mathbf{u}_1 - 2\mathbf{u}_2$ belongs to the kernel of T.

20. Fill in each blank with **must**, **might** or **cannot**, as appropriate.

a. The non-pivot columns of a matrix A ____ be linearly dependent.

b. If A is a 5×8 matrix and $\mathrm{rank}(A) = 5$, then the linear transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$ ____ be surjective and ____ be injective.

c. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent vectors in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ ____ be linearly independent.

d. The columns of an elementary matrix ____ be linearly independent.

e. If $\mathrm{Col}(A) = \mathrm{Col}(A^T)$ for an $n \times n$ matrix A, then A ____ be a symmetric matrix.

f. Given an $n \times n$ matrix A. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the equation $A\mathbf{x} = \mathbf{0}$ ____ have non-trivial solutions.

1. a. Since

$$\begin{pmatrix} A & \mathbf{b} \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 4 & 1 & 6 & 2 \\ 0 & 0 & -3 & -3 & 6 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & -3 & 14 & -10 \\ 0 & 0 & 1 & 1 & -2 & 3 \end{pmatrix},$$

the solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{q} + \operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, where

$$\mathbf{q} = \begin{pmatrix} -10 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -14 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

b. The solution of $A\mathbf{x} = \mathbf{d}$ is $\mathbf{p} + \operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, where $\mathbf{p}$ is the given particular solution of $A\mathbf{x} = \mathbf{d}$ and $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the vectors from part a.

2. Balancings of the chemical equation correspond to vectors with positive integer entries in the null space of

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 3 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{which is} \quad \operatorname{Span}\left\{\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}\right\}.$$

So the given chemical equation balances as $2KClO_3 \to 2KCl + 3O_2$.

3. Reducing $\begin{pmatrix} A & I_3 \end{pmatrix}$ gives

$$\begin{aligned} \begin{pmatrix} A & I_3 \end{pmatrix} &\sim \begin{pmatrix} -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} -1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 6 & 1 & -3 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 6 & 1 & -3 \end{pmatrix}, \end{aligned}$$

so

$$A^{-1} = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 0 & 1 \\ 6 & 1 & -3 \end{pmatrix} \quad \text{and} \quad (A^T)^{-1} = \begin{pmatrix} 3 & -2 & 6 \\ 0 & 0 & 1 \\ -2 & 1 & 3 \end{pmatrix}.$$

4. The multilinearity of the determinant and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 8$ implies that

$$\begin{vmatrix} 2a+3b & abd - b^2c \\ 2c+3d & ad^2 - bcd \end{vmatrix} = (ad-bc)\begin{vmatrix} 2a+3b & b \\ 2c+3d & d \end{vmatrix} = 2\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 = 128.$$

5. An LU factorization of A is

$$\begin{pmatrix} 3 & -1 & 1 \\ 9 & 2 & 5 \\ -6 & 22 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{via} \quad \begin{smallmatrix} 5 & 2 \\ 20 & 12 \end{smallmatrix} \rightsquigarrow 4.$$

An elementary factorization of L is

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}.$$

6. Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{pmatrix}$, in which $\mathbf{a}_2 = \mathbf{u}$ and $\mathbf{a}_5 = \mathbf{v}$. Since A is row equivalent to R, follows that

$$\mathbf{u} = 3\mathbf{a}_1 = \begin{pmatrix} 3 \\ 6 \\ 9 \\ -12 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = 2\mathbf{a}_1 + \mathbf{a}_3 + 2\mathbf{a}_4 = \begin{pmatrix} 6 \\ 11 \\ 3 \\ -2 \end{pmatrix}.$$

For the same reason, $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_4$ (the pivot columns of A) form a basis of the column space of A. Since the column space of A contains a non-zero vector, it contains infinitely many (continuum many, to be precise) vectors. The vectors

$$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix}$$

form a basis of the null space of A. The vector $\mathbf{w}$ belongs to the null space of A^T if, and only if, it is orthogonal to the (pivot) columns of A. Since $\mathbf{w}^T\mathbf{a}_1 = 16 - 4k$ it is necessary that $k = 4$; in that case $\mathbf{w}^T\mathbf{a}_3 = \mathbf{w}^T\mathbf{a}_4 = 0$, so it is also sufficient. Since $\dim \operatorname{Nul}(A^T) = 4 - \operatorname{rank}(A) = 1$, it follows that $\operatorname{Nul}(A^T)$ is a line containing the origin.

7. If C and $X + A$ are invertible, the equation $B(X+A)^{-1} = C$ is equivalent to $C^{-1}B = X + A$, or $X = C^{-1}B - A$.

8. As L is square and unit triangular $\det(L) = 1$, and $\det(A) = \det(L)\det(U)$ by product preservation, so $\det(U) = \det(A) = -3$. From multilinearity, product preservation and invariance under transposition, it follows that $\det\big(2(A^T)^3A^{-1}\big) = 2^4(-3)^3/(-3) = 144$. By multilinearity and product preservation $\det(LA + A) = \det(L + I_4)\det(A) = 2^4(-3) = 48$, since $L + I_4$ is triangular with each diagonal entry equal to 2.

9. If A is an $n \times n$ matrix whose null space has dimension zero, then $A\mathbf{x} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$. If $A^2\mathbf{x} = \mathbf{0}$, *i.e.*, $A(A\mathbf{x}) = \mathbf{0}$ then, by two applications of the preceding implication, $\mathbf{x} = \mathbf{0}$. Therefore, the dimension of the null space of A^2 is zero, as required.

10. If

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{then} \quad A + I = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so $\det(A) = \det(A + I) = 0$ (*i.e.*, both matrices are singular) as required.

11. A 2×2 matrix $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix}$ belongs to $\mathscr{H}$ if, and only if, $\mathbf{a}_1 = \mathbf{a}_2$. So

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

either of which can serve as an example of a non-zero matrix in $\mathscr{H}$, form a basis of $\mathscr{H}$.

12. Observe that $M^2 = N$ and $NM = MN = M^3 = I$, so if X, Y are linear combinations of I, M, N then XY is a linear combination of I, M, N; *i.e.*, if $X, Y \in \mathscr{W}$ then $XY \in \mathscr{W}$. Since $\mathscr{V}$ is a subspace of $\mathscr{W}$, $A \in \mathscr{V}$ implies that $A^2 \in \mathscr{W}$. Now I, M, N are linearly independent (their first columns are, respectively, $\mathbf{e}_1$, $\mathbf{e}_3$, $\mathbf{e}_2$), so $\mathscr{V}$ is a two dimensional subspace of $\mathscr{W}$ and $\mathscr{W}$ is not contained in $\mathscr{V}$, so $\mathscr{W}$ is not a (three dimensional) subspace of $\mathscr{V}$.

13. a. If $w = x = y = z = 0$ then $w = z$ and $xy = z^2$, so the zero vector of $\mathbb{R}^4$ does belong to $\mathscr{V}$.

b. The vectors $\mathbf{e}_2$ is a non-zero vector in $\mathbb{R}^4$ which belongs to $\mathscr{V}$.

c. If $w = z$, $xy = z^2$ and $\alpha \in \mathbb{R}$, then $\alpha w = \alpha z$ and $(\alpha x)(\alpha y) = \alpha^2 xy = (\alpha z)^2$, which shows that $\mathscr{V}$ is closed under scalar multiplication.

d. The vectors $\mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^4$ belong to $\mathscr{V}$, but $\mathbf{e}_2 + \mathbf{e}_3 \notin \mathscr{V}$, since $1 \cdot 1 \neq 0$. So $\mathscr{V}$ is not closed under addition.

e. Since $\mathscr{V}$ is not closed under addition, $\mathscr{V}$ is not a subspace of $\mathbb{R}^4$.

14. Let

$$\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \tfrac{1}{3}\mathbf{u} \times \mathbf{v} = \tfrac{1}{3}\begin{pmatrix} 3 \\ 9 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.$$

Then $\mathbf{u}^T\mathbf{v} = 2 > 0$ so $\angle BAC$ is an acute angle. Also, $\mathbf{n}$ is normal to the plane $\mathscr{P}$ which contains $P(1, 0, -1)$ and is parallel to the plane containing $\triangle ABC$; so $\mathscr{P}$ is defined by $\mathbf{n}^T\mathbf{x} = 3$, or $x + 3y - 2z = 3$.

15. Let

$$\mathbf{p} = \begin{pmatrix} 7 \\ 10 \\ 3 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \mathbf{p} - \mathbf{q} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}.$$

The distance between $\mathbf{p}$ and $\mathbf{q} + \operatorname{Span}\{\mathbf{v}\}$ is equal to the length of

$$\operatorname{perp}_{\mathbf{v}} \mathbf{u} = \mathbf{u} - \frac{\mathbf{v}^T\mathbf{u}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}, \quad \text{which is} \quad \tfrac{1}{2}\sqrt{54} = \tfrac{3}{2}\sqrt{6}.$$

16. Let $\ell = \mathbf{p} + \operatorname{Span}\{\mathbf{u}\}$ and $\mathscr{P} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{n}^T\mathbf{x} = -8\}$, where

$$\mathbf{p} = \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

The distance between $\mathbf{p} + t\mathbf{u}$ and the plane $\mathscr{P}$ defined by $\mathbf{n}^T\mathbf{x} = -8$ is

$$\frac{|\mathbf{n}^T(\mathbf{p} + t\mathbf{u}) + 8|}{\|\mathbf{n}\|} = \frac{4|t-2|}{3},$$

which is equal to 1 if, and only if, $t - 2 = \pm\frac{3}{4}$. Therefore, the points on ℓ ad unit distance from $\mathscr{P}$ are

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} + \frac{11}{4}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 6 \\ 23 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} + \frac{5}{4}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -6 \\ 17 \\ 10 \end{pmatrix}.$$

Since $\mathbf{n}^T\mathbf{p} = 0$ and $\mathbf{n}^T\mathbf{u} = -4$, intersection of ℓ and $\mathscr{P}$ is

$$\mathbf{p} + 2\mathbf{u} = \begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}.$$

17. The transformation T maps

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \end{pmatrix} \rightsquigarrow \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix} \rightsquigarrow \begin{pmatrix} \sin\vartheta \\ -\cos\vartheta \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore, the standard matrix of T is $-I_2$.

18. a. Since $T(\mathbf{e}_1 - \mathbf{e}_2) = T(2e_1 + 3\mathbf{e}_2)$, T is not injective.

b. By inspection (or by solving $\alpha + \beta = 2$ and $\alpha - \beta = 3$ for α, β)

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

c. By part b and

$$\frac{5}{2}T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -27 \\ 13 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -17 \\ -8 \end{pmatrix} \neq \begin{pmatrix} -27 \\ 13 \end{pmatrix} = T\begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

it follows that T is not a linear transformation.

19. If $T(\mathbf{u}_1) = T(\mathbf{u}_2)$ and T is linear, $T(2\mathbf{u}_1 - 2\mathbf{u}_2) = 2\big(T(\mathbf{u}_1) - T(\mathbf{u}_2)\big) = \mathbf{0}$, so $2\mathbf{u}_1 - 2\mathbf{u}_2$ belongs to the kernel of T.

20. a. The non-pivot columns of a matrix A **might** be linearly dependent.

b. If A is a 5×8 matrix and $\operatorname{rank}(A) = 5$, then the linear transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$ **must** be surjective and **cannot** be injective.

c. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent vectors in $\operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ **must** be linearly independent.

d. The columns of an elementary matrix **must** be linearly independent.

e. If $\operatorname{Col}(A) = \operatorname{Col}(A^T)$ for an $n \times n$ matrix A, then A **might** be a symmetric matrix.

f. Given an $n \times n$ matrix A. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the equation $A\mathbf{x} = \mathbf{0}$ **must** have non-trivial solutions.