1. Let $\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 5 & 3 & 0 & 1 \\ -1 & -3 & -2 & 1 & 0 \\ 0 & -1 & -1 & 2 & 1 \end{pmatrix}$ be the augmented matrix of $A\mathbf{x} = \mathbf{b}$.

a. Determine whether $\begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ is a solution of the equation.

b. Give the general solution of the equation.

c. What is the general solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$?

d. Write the fourth column of A as a linear combination of the first three columns of A.

2. Let $A = \begin{pmatrix} 1 & 1 & a \\ 1 & a & a \\ a & a & a \\ a & a & a^2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ a^2 - 2a \end{pmatrix}$.

a. For which value(s) of a does $A\mathbf{x} = \mathbf{b}$ have no solution?

b. For which value(s) of a does $A\mathbf{x} = \mathbf{b}$ have a unique solution?

c. For which value(s) of a does $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions?

3. Find a quadratic polynomial whose graph contains the points $(-1,3)$, $(0,3)$ and $(1,7)$.

4. Find the inverse of $\begin{pmatrix} 1 & -1 & -8 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 2 \end{pmatrix}$.

a. Evaluate A^TA and $(A^TA)^{-1}$.

b. Evaluate AA^T and explain why AA^T is not invertible.

6. Find an LU factorization of $\begin{pmatrix} -2 & -2 \\ -4 & -1 \\ -10 & 2 \end{pmatrix}$.

7. Let $A = \begin{pmatrix} 5 & 6 \\ 3 & 2 \end{pmatrix}$.

a. Apply an elementary row operation to A so that the resulting matrix is lower triangular, and write the corresponding elementary matrix.

b. Use part a to find an upper triangular matrix U and a lower triangular matrix L such that $A = UL$.

8. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and let $T(\mathbf{x}) = A\mathbf{x}$.

a. Find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 2 \end{pmatrix}$.

b. Find bases for the kernel and range of T.

c. Is T surjective? Is T injective?

9. Let $\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

a. Find a 2×2 rotation matrix A such that $A\mathbf{u}$ is orthogonal to $\mathbf{u}$.

b. Find the standard matrix B of a horizontal shear so that $B\mathbf{u}$ is orthogonal to $\mathbf{u}$.

c. Draw $\mathbf{u}$ and $AB\mathbf{u}$.

10. Expand and simplify $\begin{pmatrix} -A^{-1} & BA^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} AB & B \\ A & 0 \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ B^{-1} & A \end{pmatrix}$.

11. Let $A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 0 \end{pmatrix}$.

a. Find $\det(A)$.

b. What is $\det(-2A^{-1}A^TA)$?

12. Suppose that A, B and C are $n \times n$ matrices such that $ABCA = I$.

a. Use determinants to explain why A, B and C are invertible.

b. Express C^{-1} in terms of A and B (in simplest form).

13. Find the rank and nullity of each matrix A described below.

a. A is a 5×5 elementary matrix.

b. A is the standard matrix of a linear transformation of $\mathbb{R}^7$ onto $\mathbb{R}^5$.

c. A is a non-zero 2×2 matrix such that A^2 is the zero 2×2 matrix.

14. Consider the set $\mathscr{H} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : abcd = 0 \right\}$.

a. Is $\mathscr{H}$ closed under scalar multiplication? Justify your answer.

b. Is $\mathscr{H}$ closed under addition? Justify your answer.

15. Let $\mathscr{V} = \{p \in \mathbb{P}_2 : p(0) = -p'(1)\}$ and $q(x) = 6x + k$, where $k \in \mathbb{R}$.

a. Find a basis of $\mathscr{V}$.

b. For what value of k is $q \in \mathscr{V}$?

c. Is the derivative $q' \in \mathscr{V}$? Is the second derivative $q'' \in \mathscr{V}$?

16. Let $\ell_1 = \begin{pmatrix} 0 \\ -6 \\ -4 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right\}$ and $\ell_2 = \begin{pmatrix} 7 \\ -2 \\ 9 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \right\}$.

a. Find the point of intersection of ℓ_1 and ℓ_2.

b. Give a cartesian equation (of the form $ax+by+cz = d$) of the plane which is orthogonal to ℓ_1 and contains $Q(2,1,1)$.

c. Find the cosine of the acute angle between ℓ_1 and ℓ_2.

17. Consider the triangular prism in $\mathbb{R}^3$ whose base has vertices $A(0,1,3)$, $B(2,-1,3)$ and $C(1,1,5)$, and whose other vertex adjacent to A is $D(4,7,10)$.

a. Find a parametric vector equation of the line containing A and B.

b. Find the area of $\triangle ABC$.

c. Find the volume of the prism.

18. Simplify $\text{proj}_{\mathbf{u}+\mathbf{w}}(\mathbf{u} - 2\mathbf{v})$, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal unit vectors in $\mathbb{R}^n$.

19. Let A be an $m \times n$ matrix for which there is an matrix C such that $AC = I_m$. Show that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$. What can you conclude about the rank of A?

20. Fill in the blanks with **must**, **might** or **cannot**.

a. If $A^2 + 3A = 2I$ then A ____ be invertible.

b. If A, B and C are points in $\mathbb{R}^n$ such that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$, then $\text{Span}\{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}\}$ ____ be three-dimensional.

c. Two lines in $\mathbb{R}^3$ which are orthogonal to a third line ____ be parallel.

d. If $\mathbf{a}$, $2\mathbf{a}+3\mathbf{b}$, $\mathbf{a}-3\mathbf{c}$ are linearly independent vectors in a vector space V, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ ____ be linearly independent.

1. a. Since

$$\begin{pmatrix}1 & 2 & 1 & 1\\ 2 & 5 & 3 & 0\\ -1 & -3 & -2 & 1\\ 0 & -1 & -1 & 2\end{pmatrix}\begin{pmatrix}4\\-2\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$$

the vector in question is a solution of $A\mathbf{x}=\mathbf{b}$.

b. Writing $A=\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4\end{pmatrix}$; then $\mathbf{a}_1$ and $\mathbf{a}_2$ are the pivot columns of A, $\mathbf{a}_3=-\mathbf{a}_1+\mathbf{a}_2$, $\mathbf{a}_4=5\mathbf{a}_1-2\mathbf{a}_2$ and $\mathbf{b}=3\mathbf{a}_1-\mathbf{a}_2$. Therefore, the solution of $A\mathbf{x}=\mathbf{b}$ is $\mathbf{p}+\operatorname{Span}\{\mathbf{u},\mathbf{v}\}$, where

$$\mathbf{p}=\begin{pmatrix}3\\-1\\0\\0\end{pmatrix},\qquad \mathbf{u}=\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}\qquad\text{and}\qquad \mathbf{v}=\begin{pmatrix}-5\\2\\0\\1\end{pmatrix}.$$

c. The solution of $A\mathbf{x}=\mathbf{0}$ is $\operatorname{Span}\{\mathbf{u},\mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are from part b.

d. The fourth column of A is given by $\mathbf{a}_4=5\mathbf{a}_1-2\mathbf{a}_2$, as was observed already in part a.

2. Reducing $\begin{pmatrix}A & \mathbf{b}\end{pmatrix}$ gives

$$\begin{pmatrix}A & \mathbf{b}\end{pmatrix}\sim\begin{pmatrix}1 & 1 & a & 1\\ 0 & a-1 & 0 & -1\\ 0 & 0 & a-a^2 & -a\\ 0 & 0 & 0 & a^2-3a\end{pmatrix}.$$

The equation $A\mathbf{x}=\mathbf{b}$ has no solution if, and only if, $\mathbf{b}$ is a pivot column of the matrix $\begin{pmatrix}A & \mathbf{b}\end{pmatrix}$, which occurs if, and only if, a is neither 0 nor 3. The equation has a unique solution if, and only if, $\mathbf{b}$ is not a pivot column of $\begin{pmatrix}A & \mathbf{b}\end{pmatrix}$ and A has three pivot columns, which occurs if, and only if, $a=3$. The equation has infinitely many solutions if, and only if $\mathbf{b}$ is not a pivot column of $\begin{pmatrix}A & \mathbf{b}\end{pmatrix}$ and A has at most two pivot columns, which occurs if, and only if $a=0$.

3. The coefficients (in order of increasing degree) of the polynomial are the entries of the solution of the equation whose augmented matrix is

$$\begin{pmatrix}1 & 0 & 0 & 3\\ 1 & 1 & 1 & 7\\ 1 & -1 & 1 & 3\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 0 & 3\\ 0 & 1 & 1 & 4\\ 0 & -1 & 1 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 0 & 3\\ 0 & 1 & 1 & 4\\ 0 & 0 & 2 & 4\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 0 & 3\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 2\end{pmatrix}.$$

Therefore, the polynomial in question is $3+2x+2x^2$.

4. Applying to I_3 the elementary row operations which reduce the given matrix to I_3 gives

$$I_3\sim\begin{pmatrix}1 & 0 & 8\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}\sim\begin{pmatrix}1 & 1 & 10\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}=A^{-1}.$$

5. If

$$A=\begin{pmatrix}1 & 3\\ 0 & 1\\ -1 & 2\end{pmatrix},\quad\text{then}\quad A^TA=\begin{pmatrix}2 & 1\\ 1 & 14\end{pmatrix}\quad\text{and}\quad (A^TA)^{-1}=\tfrac{1}{27}\begin{pmatrix}14 & -1\\ -1 & 2\end{pmatrix}.$$

The rank of AA^T is at most the rank of A, which is 2, so

$$AA^T=\begin{pmatrix}10 & 3 & 5\\ 3 & 1 & 2\\ 5 & 2 & 5\end{pmatrix}$$

is not inverible.

6. An LU factorization of the given matrix is

$$\begin{pmatrix}-2 & -2\\ -4 & -1\\ -10 & 2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 5 & 4 & 1\end{pmatrix}\begin{pmatrix}-2 & -2\\ 0 & 3\\ 0 & 0\end{pmatrix}\qquad\text{via}\qquad \substack{3\\12}\rightsquigarrow 0.$$

7. Adding -3 times the second row of A to the first row gives

$$\begin{pmatrix}1 & -3\\ 0 & 1\end{pmatrix}\begin{pmatrix}5 & 6\\ 3 & 2\end{pmatrix}=\begin{pmatrix}-4 & 0\\ 3 & 2\end{pmatrix},$$

or

$$\begin{pmatrix}5 & 6\\ 3 & 2\end{pmatrix}=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}-4 & 0\\ 3 & 2\end{pmatrix}.$$

8. By inspection

$$\begin{pmatrix}1 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}4\\2\\0\end{pmatrix}=\begin{pmatrix}4\\4\\2\\2\end{pmatrix},$$

the first and second columns of A form a basis of the range of T and $\mathbf{e}_1-\mathbf{e}_3$ forms a basis of the kernel of T, so T is neither injective nor surjective.

9. Any rotation through an odd multiple of a right angle will map $\mathbf{u}$ to a vector which is orthogonal to $\mathbf{u}$; for example if

$$A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\qquad\text{then}\qquad A\mathbf{u}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}-2\\-1\end{pmatrix}$$

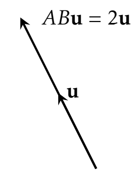
is orthogonal to $\mathbf{u}$, and A is the standard matrix of a rotation about the origin by $\frac{1}{2}\pi$. For the shear, it is required to find α so that

$$\begin{pmatrix}-1 & 2\end{pmatrix}\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}=0,\qquad\text{or}\qquad 5-2\alpha=0,$$

so if

$$B=\begin{pmatrix}1 & \frac{5}{2}\\ 0 & 1\end{pmatrix},\qquad\text{then}\qquad B\mathbf{u}=\begin{pmatrix}1 & \frac{5}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$$

is orthogonal to $\mathbf{u}$. For these A and B, $AB\mathbf{u}=\begin{pmatrix}-2\\4\end{pmatrix}=2\mathbf{u}$. Below is a picture.



10. A direct computation gives

$$\begin{pmatrix}-A^{-1} & BA^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}AB & B\\ A & 0\end{pmatrix}\begin{pmatrix}A^{-1} & 0\\ B^{-1} & A\end{pmatrix}=\begin{pmatrix}-A^{-1} & BA^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}ABA^{-1}+I & BA\\ I & 0\end{pmatrix}$$
$$=\begin{pmatrix}-A^{-1} & -A^{-1}BA\\ I & 0\end{pmatrix}.$$

11. A direct computation (reduction and Laplace expansion) gives

$$\det(A)=\begin{vmatrix}0 & 1 & 2 & 1\\ 1 & 0 & 2 & 1\\ 1 & 2 & 0 & 1\\ 1 & 2 & 4 & 0\end{vmatrix}=\begin{vmatrix}1 & 2 & 1\\ -2 & 2 & 0\\ 0 & 4 & -1\end{vmatrix}=\begin{vmatrix}6 & 2\\ 4 & -1\end{vmatrix}=-14.$$

Multilinearity, product preservation and invaiance under transposition then implies that $\det(-2A^{-1}A^TA)=(-2)^4(-14)^{-1}(-14)^2=-16\cdot 14=-224$.

12. For $n\times n$ matrices A, B, C such that $ABCA=I$, product preservation of the determinant implies that $\det(A)^2\det(B)\det(C)=\det(ABCA)=1$, so that $\det(A)$, $\det(B)$, $\det(C)$ are non-zero, and thus A, B, C are invertible.

The determinant is of course unnecessary, for if X and Y are square matrices of the same size and $XY=I$, then $YX=I$ and X, Y are inverses. Hence, for for $n\times n$ matrices A, B, C, the equation $ABCA=I$ implies that $A^{-1}=ABC=BCA$, $B^{-1}=CA^2$, and $C^{-1}=A^2B$, which answers (more than) the second part of the question.

13. A 5×5 elementary matrix is invertible, so its rank is 5 and its nullity is 0. The column space of the standard matrix of a surjective linear transformation $\mathbb{R}^7 \to \mathbb{R}^5$ is $\mathbb{R}^5$, so its rank is 5 and its nullity is $7-5=2$. If A is a non-zero 2×2 matrix then the rank of A is at least 1, and if $A^2=0$ then the nullity of A is at least 1; since these add up to 2, the rank and nullity of A must each be equal to 1.

14. If

$$A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\mathscr{H}, \qquad \text{and } \alpha\in\mathbb{R}, \text{ then } \qquad \alpha A=\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d\end{pmatrix},$$

and $(\alpha a)(\alpha b)(\alpha c)(\alpha d)=\alpha^4 abcd=\alpha^4 0=0$, so $\alpha A\in\mathscr{H}$. So $\mathscr{H}$ closed under scalar multiplication. On the other hand,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\in\mathscr{H}, \qquad \text{but} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}+\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}\notin\mathscr{H}$$

(for $1\cdot 0\cdot 1\cdot 1=0\cdot 1\cdot 1\cdot 0=0$ but $1\cdot 1\cdot 1\cdot 1\neq 0$), so $\mathscr{H}$ is not closed under addition.

15. If $p(x)=a+bx+cx^2$ then $p(0)=a$ and $-p'(1)=-b-2c$, so $p\in\mathscr{V}$ if, and only if, $a=-b-2c$, or $p(x)=b(x-1)+c(x^2-2)$. Therefore $\{x-1, x^2-2\}$ is a basis for $\mathscr{V}$. It follows that $q(x)=6x+k\in\mathscr{V}$ if, and only if, it is a scalar multiple of $x-1$, or $k=-6$. The derivative $q'(x)=6\notin\mathscr{V}$, since $6=q'(0)\neq -q''(1)=0$. On the other hand, it is plain that the second derivative $q''(x)=0\in\mathscr{V}$.

16. Let

$$\mathbf{p}_1=\begin{pmatrix}0\\-6\\-4\end{pmatrix}, \qquad \mathbf{p}_2=\begin{pmatrix}7\\-2\\9\end{pmatrix}, \qquad \mathbf{v}_1=\begin{pmatrix}-1\\2\\3\end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2=\begin{pmatrix}4\\1\\5\end{pmatrix}.$$

The equation $\mathbf{p}_1+t_1\mathbf{v}_1=\mathbf{p}_2=t_2\mathbf{v}_2$ is solved for t_1 and $-t_2$ by reducing

$$\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{p}_2-\mathbf{p}_1\end{pmatrix}\sim\begin{pmatrix}-1 & 4 & 7\\ 0 & 9 & 18\\ 0 & 17 & 34\end{pmatrix}\sim\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix},$$

so the point of intersection of ℓ_1 and ℓ_2 is

$$\mathbf{p}_1+\mathbf{v}_1=\mathbf{p}_2-2\mathbf{v}_2=\begin{pmatrix}-1\\-4\\-1\end{pmatrix}.$$

The vector $\mathbf{v}_1$ is a normal to any plane which is orthogonal to ℓ_1, so the plane in question is defined by $\mathbf{v}_1^T\mathbf{x}=\mathbf{v}_1^T\,\overrightarrow{OQ}$, or $x-2y-3z=-3$ (taking x, y and z to be the entries of $\mathbf{x}$). The cosine of the acute anngle between ℓ_1 and ℓ_2 is equal to

$$\frac{|\mathbf{v}_1^T\mathbf{v}_2|}{\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|}=\frac{13}{\sqrt{14}\sqrt{42}}=\tfrac{13}{42}\sqrt{3}.$$

17. Let

$$\mathbf{a}=\overrightarrow{OA}=\begin{pmatrix}0\\1\\3\end{pmatrix}, \ \mathbf{u}=\overrightarrow{AB}=\begin{pmatrix}2\\-2\\0\end{pmatrix}, \ \mathbf{v}=\overrightarrow{AC}=\begin{pmatrix}1\\0\\2\end{pmatrix} \ \text{and} \ \mathbf{w}=\overrightarrow{AD}=\begin{pmatrix}4\\6\\7\end{pmatrix}.$$

The line containing A and B is $\mathbf{a}+\operatorname{Span}\{\mathbf{u}\}$. The area of $\triangle ABC$ is the length of

$$\tfrac{1}{2}\mathbf{u}\times\mathbf{v}=\tfrac{1}{2}\begin{pmatrix}-4\\-4\\2\end{pmatrix}=-\begin{pmatrix}2\\2\\-1\end{pmatrix},$$

or 3. The volume of the prism is $\left|\tfrac{1}{2}(\mathbf{u}\times\mathbf{v})^T\mathbf{w}\right|=13$.

18. Since $(\mathbf{u}+\mathbf{w})^T(\mathbf{u}-2\mathbf{v})=\mathbf{u}^T\mathbf{u}=1$ and $(\mathbf{u}+\mathbf{w})^T(\mathbf{u}+\mathbf{w})=\mathbf{u}^T\mathbf{u}+\mathbf{w}^T\mathbf{w}=2$, for mutually orthogonal unit vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\operatorname{proj}_{\mathbf{u}+\mathbf{w}}(\mathbf{u}-2\mathbf{v})=\tfrac{1}{2}(\mathbf{u}+\mathbf{w})$.

19. If $AC=I_m$ and $\mathbf{b}\in\mathbb{R}^m$, then $A(C\mathbf{b})=(AC)\mathbf{b}=I_m\mathbf{b}=\mathbf{b}$. Therefore, the column space of A is $\mathbb{R}^m$ and the rank of A is m.

20. a. If $A^2+3A=2I$ then A **must** be invertible.

b. If A, B and C are points in $\mathbb{R}^n$ such that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\mathbf{0}$, then $\operatorname{Span}\{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}\}$ **cannot** be three-dimensional.

c. Two lines in $\mathbb{R}^3$ which are orthogonal to a third line **might** be parallel.

d. If $\mathbf{a}$, $2\mathbf{a}+3\mathbf{b}$, $\mathbf{a}-3\mathbf{c}$ are linearly independent vectors in a vector space V, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ **must** be linearly independent.