

Question 1. — You are given the matrix A and its reduced echelon form R :

$$A = \begin{pmatrix} 0 & 0 & 2 & 4 & 0 \\ 2 & 2 & 0 & 6 & 4 \\ 1 & 1 & 2 & 7 & 2 \end{pmatrix} \sim R = \begin{pmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Column j of A will be denoted by \mathbf{a}_j .

- Write the general solution of $A\mathbf{x} = \mathbf{0}$.
- Find the dimension of $\text{Nul}(A)$.
- Give a basis of $\text{Col}(A)$.

d. For which value of a is $\begin{pmatrix} 6 \\ a \\ 0 \end{pmatrix}$ in $\text{Col}(A)$?

e. Which of the following is a basis of $\text{Col}(A)$? Justify your answers briefly.

$$- \{\mathbf{a}_2, \mathbf{a}_4\} \quad - \{\mathbf{a}_2, \mathbf{a}_5\} \quad - \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$$

Question 2. — You are given the following system:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ x_1 + x_2 + ax_3 &= 6 \\ x_1 + 4x_2 - x_3 &= b \end{aligned}$$

- For what values of a and b is this system consistent?
- For what values of a and b does this system have a unique solution?
- For what values of a and b does this system have infinitely many solutions?

d. For what values of a and b is $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ a solution of this system?

Question 3. — Find an equation of the form $y = a + bx + cx^2$ whose graph contains the points $(-1, 1)$, $(1, 5)$ and $(2, 1)$.

Question 4. — a. Find a (non-zero) vector \mathbf{u} in \mathbb{R}^2 which is parallel to the line $y = -2x$.

b. Find a (non-zero) vector \mathbf{v} in \mathbb{R}^2 which is perpendicular to the line $y = -2x$.

c. Write $\begin{pmatrix} 5 \\ 15 \end{pmatrix}$ as a linear combination of the vectors \mathbf{u} and \mathbf{v} .

Question 5. — Let $A = \begin{pmatrix} 3 & a \\ 1 & b \end{pmatrix}$.

- For what values of a and b (if any) is A symmetric?
- For what values of a and b (if any) is $A^2 = A$?
- For what values of a and b (if any) is $A = A^{-1}$?
- Find a condition on a and b such that A is invertible.

Question 6. — Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

- Compute A^2 .
- Based on the answer in part a, what is A^{-1} ?

Question 7. — Let $S: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation which reflects vectors through the x_2 axis and let $R: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation which reflects vectors through the line defined by $x_1 = x_2$.

- Find the standard matrix of $R \circ S$.
- Find the angle (between 0 and 2π) of rotation of $R \circ S$.

Question 8. — Let \mathbf{a} , \mathbf{b} and \mathbf{c} be vectors in \mathbb{R}^2 , and let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an invertible linear transformation such that $T(\mathbf{a} + \mathbf{b}) = \mathbf{c}$, $T(\mathbf{a} + \mathbf{c}) = \mathbf{b}$ and $T(\mathbf{b} + \mathbf{c}) = \mathbf{a}$.

- What is $T^{-1}(\mathbf{a})$?

b. What is $T(\mathbf{a} + \mathbf{b} + \mathbf{c})$ in its simplest form?

c. What is $T(\mathbf{a})$?

Question 9. — Write $\begin{pmatrix} 1 & 1 \\ 1 & 100 \end{pmatrix}$ as a product of elementary matrices.

Question 10. — Suppose that A and B are invertible $n \times n$ matrices. Find the inverse of

$$\begin{pmatrix} A & I \\ B^{-1}A & 0 \end{pmatrix}.$$

Question 11. — Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 7 \end{pmatrix}$.

- Evaluate the determinant of A .
- Use Cramer's rule to solve $A\mathbf{x} = 2\mathbf{e}_3$ for x_4 only.

Question 12. — Let $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and $V = \{\mathbf{x} \in \mathbb{R}^2: \mathbf{x} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{v}\}$.

- Find a basis of V .
- Draw a picture of V .

Question 13. — For $n \times n$ matrices A and B , let

$$V = \{X \in M_{n \times n}: AXB = BXA\}.$$

- Show that V is a subspace of $M_{n \times n}$.
- Show that if A is invertible then $A^{-1} \in V$.

Question 14. — Let $S = \{1 + 2x - x^2, 1 + 3x^2, 4x + ax^2\}$.

- For which values of a is S linearly dependent?
- What is the dimension of $\text{Span } S$ if $a = 0$?

Question 15. — Let

$$\ell_1 = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \ell_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

- Find the distance between ℓ_1 and ℓ_2 .
- Find an equation of the line through the origin which intersects ℓ_1 at a right angle.

Question 16. — Let T be the triangle with vertices $A(1, 1, 1)$, $B(2, 2, 2)$ and $C(3, -1, 2)$.

- Find a normal equation of the plane containing T .
- Find the area of T .
- Find the cosine of the angle at vertex A of T .
- Find the point on side AC which is 2 units from A .

Question 17. — Let $\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

- Compute $\mathbf{e}_1 \times \mathbf{w}$, $\mathbf{e}_2 \times \mathbf{w}$ and $\mathbf{e}_3 \times \mathbf{w}$.
- Express $\|\mathbf{e}_1 \times \mathbf{w}\|^2 + \|\mathbf{e}_2 \times \mathbf{w}\|^2 + \|\mathbf{e}_3 \times \mathbf{w}\|^2$ in terms of $\|\mathbf{w}\|$.

Question 18. — Fill in the correct numerical value for each of the following statements.

a. If A is a 7×4 matrix of rank 3 and $A\mathbf{x} = \mathbf{b}$ is consistent, then the rank of the 7×5 matrix $\begin{pmatrix} A & \mathbf{b} \end{pmatrix}$ is ____.

b. If $\left\{ \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ a \\ b \end{pmatrix} \right\}$ is linearly dependent, then $a =$ ____ and $b =$ ____.

c. If $\det(A) = 5$ and $\det(2A) = 40$ then $\det(3A) =$ ____.

d. If A is a 6×6 matrix such that $\text{Row}(A)$, $\text{Col}(A)$ and $\text{Nul}(A)$ all have the same dimension, then the rank of A is ____.

Solution to Question 1. — a. By inspecting R , the general solution of $Ax = 0$ is

$$\text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

- b. The dimension of $\text{Nul}(A)$ is 3.
- c. The vector in question must be $-6\mathbf{a}_1 + 3\mathbf{a}_2$, so $a = -12$.
- d. Only $\{\mathbf{a}_2, \mathbf{a}_4\}$ is a basis of $\text{Col}(A)$ as it is the only list of two linearly independent vectors (and the rank of A is 2).

Solution to Question 2. — Reducing the augmented matrix of the system gives

$$(A \ \mathbf{b}) \sim \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 3 \\ 0 & -2 & a-1 & 6 \\ 0 & 1 & -2 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & a-5 & 0 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$$

- a. The system is inconsistent if $b \neq -3$, regardless of the value of a .
- b. The system has a unique solution if $b = -3$ and $a \neq 5$.
- c. The system has infinitely many solutions if $b = -3$ and $a = 5$.
- d. The system is inconsistent unless $b = -3$, in which case the third entry of the solution must be zero unless $a = 5$. For these values of a and b the given vector is indeed a solution of the equation.

Solution to Question 3. — The coefficients of the required polynomial form the solution of $Ax = \mathbf{b}$, where

$$(A \ \mathbf{b}) = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & 4 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -4 \\ 0 & -2 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 6 & -12 \end{pmatrix} \\ \sim \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Thus, the required polynomial is $5 + 2x - 2x^2$.

Solution to Question 4. — a. The vector $\mathbf{u} = \mathbf{e}_1 - 2\mathbf{e}_2$ is a direction vector to the line defined by $y = -2x$ (so it is parallel to this line).

b. The vector $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2$ is orthogonal to \mathbf{u} , so it is orthogonal to the line defined by $y = -2x$.

c. Writing \mathbf{x} for the vector in question, the orthogonality of \mathbf{u}, \mathbf{v} gives

$$\begin{pmatrix} 5 \\ 15 \end{pmatrix} = \mathbf{x} = \frac{\mathbf{u}^T \mathbf{x}}{\mathbf{u}^T \mathbf{u}} \mathbf{u} + \frac{\mathbf{v}^T \mathbf{x}}{\mathbf{v}^T \mathbf{v}} \mathbf{v} = -5 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution to Question 5. — a. The matrix A is symmetric (i.e., $A^T = A$) if, and only if, $a = 1$ (b can be any real number).

b. Since

$$A^2 = \begin{pmatrix} 3 & a \\ 1 & b \end{pmatrix} \begin{pmatrix} 3 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} a+9 & a(b+3) \\ b+3 & a+b^2 \end{pmatrix},$$

$A^2 = A$ if, and only if, $a = -6$ and $b = -2$.

c. From part b, $A^{-1} = A$ if, and only if, $a = -8$ and $b = -3$.

d. Since $\det(A) = 3b - a$, A is invertible if, and only if, $a \neq 3b$.

Solution to Question 6. — Since A has mutually orthogonal columns of length 2, and $A^T = A$, it follows that $A^2 = A^T A = 4I_4$ and $A^{-1} = \frac{1}{4}A$.

Solution to Question 7. — a. The standard matrices of S and R are

$$B = [S] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = [R] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so the standard matrix of $R \circ S$ is

$$[R \circ S] = AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

b. The linear transformation $R \circ S$ maps $\mathbf{e}_1 \rightsquigarrow -\mathbf{e}_2$ and $\mathbf{e}_2 \rightsquigarrow \mathbf{e}_1$, which effects a rotation about the origin through $\frac{3}{2}\pi$ (radians).

Solution to Question 8. — a. Since $T(\mathbf{b} + \mathbf{c}) = \mathbf{a}$, $T^{-1}(\mathbf{a}) = \mathbf{b} + \mathbf{c}$.

b. Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = \frac{1}{2}((\mathbf{b} + \mathbf{c}) + (\mathbf{a} + \mathbf{c}) + (\mathbf{a} + \mathbf{b}))$, it follows that

$$T(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

c. Since $T(\mathbf{b} + \mathbf{c}) = \mathbf{a}$, the result of part b gives

$$T(\mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{a} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

Solution to Question 9. — The elementary row operations $R_2 \leftarrow R_2 - R_1$, $R_2 \leftarrow \frac{1}{99}R_2$, $R_1 \leftarrow R_1 - R_2$ yield

$$\begin{pmatrix} 1 & 1 \\ 1 & 100 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 99 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} 1 & 1 \\ 1 & 100 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 99 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Solution to Question 10. — Since

$$\begin{pmatrix} A & I \\ B^{-1}A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{-1}B \\ I & -B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

it follows that

$$\begin{pmatrix} A & I \\ B^{-1}A & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & A^{-1}B \\ I & -B \end{pmatrix}.$$

Solution to Question 11. — a. By a direct computation,

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 4 & 6 \end{vmatrix} = 8 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 8.$$

b. Since

$$\det(A_4(2\mathbf{e}_3)) = 2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 2 & 4 & 0 \end{vmatrix} = 8 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -8,$$

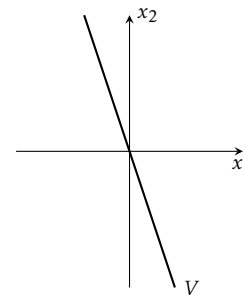
it follows that $x_4 = -1$.

Solution to Question 12. — a. Note that V is the set of all vectors $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{w}^T \mathbf{x} = 0$, where

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{so} \quad \left\{ \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$$

is a basis for V .

b. Below is a picture of V .



Solution to Question 13. — a. If $X, Y \in V$ and $\alpha, \beta \in \mathbb{R}$, then $AXB = BXA$ and $AYB = BYA$, so

$$A(\alpha X + \beta Y)B = \alpha AXB + \beta AYB = \alpha BXA + \beta BYA = B(\alpha X + \beta Y)A;$$

thus $\alpha X + \beta Y \in V$. Since V is non-empty (the $n \times n$ zero matrix belongs to V), it follows that V is a subspace of $M_{n \times n}$.

b. If A is invertible, then $AA^{-1}B = B = BA^{-1}A$, and thus $A^{-1} \in V$.

Solution to Question 14. — a. The reduction

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 4 \\ -1 & 3 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & a+8 \end{pmatrix}$$

implies that the dependence equation

$$\alpha(1+2x-x^2) + \beta(1+3x^2) + \gamma(4x+ax^2) = 0$$

has non-trivial solutions if, and only if, $a = -8$.

b. By part a S is linearly independent unless $a = -8$, so the dimension of $\text{Span } S$ is 3 if $a = 0$.

Solution to Question 15. — a. Let

$$\mathbf{u} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \text{so} \quad \mathbf{u} \times \mathbf{v} = \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix}.$$

The distance between ℓ_1 and ℓ_2 is

$$\frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{(-4)^2 + 1^2 + 7^2}}{\sqrt{2^2 + 1^2 + 1^2}} = \sqrt{11}.$$

b. The line which contains the origin and intersects ℓ_1 at a right angle is $\text{Span}\{\text{perp}_{\mathbf{v}} \mathbf{w}\}$, where \mathbf{v} is from part a and

$$\mathbf{w} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \quad \text{so} \quad \text{perp}_{\mathbf{v}} \mathbf{w} = \mathbf{w} - \frac{\mathbf{v}^T \mathbf{w}}{\mathbf{v}^T \mathbf{v}} \mathbf{v} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} - \frac{7}{3} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}.$$

Solution to Question 16. — Let

$$\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \text{so} \quad \mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}.$$

a. The plane of T is defined by $\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \overrightarrow{OA}$, or $3x_1 + x_2 - 4x_3 = 0$.

b. The area of T is $\frac{1}{2} \|\mathbf{n}\| = \frac{1}{2} \sqrt{26}$.

c. The cosine of the angle in T at A is

$$\frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1}{3\sqrt{3}} = \frac{1}{9} \sqrt{3}.$$

d. The point on side AC which is 2 units from A is

$$\overrightarrow{OA} + \frac{2}{\|\mathbf{v}\|} \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}.$$

Solution to Question 17. — a. Direct computations gives

$$\mathbf{e}_1 \times \mathbf{w} = \begin{pmatrix} 0 \\ -c \\ b \end{pmatrix}, \quad \mathbf{e}_2 \times \mathbf{w} = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} \quad \text{and} \quad \mathbf{e}_3 \times \mathbf{w} = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}.$$

b. By part a, $\|\mathbf{e}_1 \times \mathbf{w}\|^2 + \|\mathbf{e}_2 \times \mathbf{w}\|^2 + \|\mathbf{e}_3 \times \mathbf{w}\|^2 = 2(a^2 + b^2 + c^2) = 2\|\mathbf{w}\|^2$.

Solution to Question 18. — a. If A is a 7×4 matrix of rank 3 and $A\mathbf{x} = \mathbf{b}$ is consistent, then the rank of the 7×5 matrix $(A \quad \mathbf{b})$ is $\underline{3}$.

b. If $\left\{ \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ a \\ b \end{pmatrix} \right\}$ is linearly dependent, then $a = \frac{4}{3}$ and $b = -\frac{14}{3}$.

c. If $\det(A) = 5$ and $\det(2A) = 40$ then $\det(3A) = \underline{135}$.

d. If A is a 6×6 matrix such that $\text{Row}(A)$, $\text{Col}(A)$ and $\text{Nul}(A)$ all have the same dimension, then the rank of A is $\underline{3}$.