Question 1. — You are given the matrix A and its reduced echelon form R:

$$A = \begin{pmatrix} 0 & 0 & 2 & 4 & 0 \\ 2 & 2 & 0 & 6 & 4 \\ 1 & 1 & 2 & 7 & 2 \end{pmatrix} \sim R = \begin{pmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Column j of A will be denoted by $\mathbf{a}_j$.

a. Write the general solution of $A\mathbf{x} = \mathbf{0}$.
b. Find the dimension of $\mathrm{Nul}(A)$.
c. Give a basis of $\mathrm{Col}(A)$.
d. For which value of a is $\begin{pmatrix} 6 \\ a \\ 0 \end{pmatrix}$ in $\mathrm{Col}(A)$?
e. Which of the following is a basis of $\mathrm{Col}(A)$? Justify briefly.
– $\{\mathbf{a}_2, \mathbf{a}_4\}$ – $\{\mathbf{a}_2, \mathbf{a}_5\}$ – $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

Question 2. — You are given the following system:

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & x_3 & = & 0 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ x_1 & + & x_2 & + & ax_3 & = & 6 \\ x_1 & + & 4x_2 & - & x_3 & = & b \end{array}$$

a. For what values of a and b is this system consistent?
b. For what values of a and b does this system have a unique solution?
c. For what values of a and b does this system have infinitely many solutions?
d. For what values of a and b is $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ a solution of this system?

Question 3. — Find an equation of the form $y = a + bx + cx^2$ whose graph contains the points $(-1, 1)$, $(1, 5)$ and $(2, 1)$.

Question 4. — a. Find a (non-zero) vector $\mathbf{u}$ in $\mathbb{R}^2$ which is parallel to the line $y = -2x$.
b. Find a (non-zero) vector $\mathbf{v}$ in $\mathbb{R}^2$ which is perpendicular to the line $y = -2x$.
c. Write $\begin{pmatrix} 5 \\ 15 \end{pmatrix}$ as a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$.

Question 5. — Let $A = \begin{pmatrix} 3 & a \\ 1 & b \end{pmatrix}$.

a. For what values of a and b (if any) is A symmetric?
b. For what values of a and b (if any) is $A^2 = A$?
c. For what values of a and b (if any) is $A = A^{-1}$?
d. Find a condition on a and b such that A is invertible.

Question 6. — Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

a. Compute A^2. b. Based on the answer in part a, what is A^{-1}?

Question 7. — Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects vectors through the x_2 axis and let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects vectors through the line defined by $x_1 = x_2$.

a. Find the standard matrix of $R \circ S$.
b. Find the angle (between 0 and 2π) of rotation of $R \circ S$.

Question 8. — Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^2$, and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transformation such that $T(\mathbf{a}+\mathbf{b}) = \mathbf{c}$, $T(\mathbf{a}+\mathbf{c}) = \mathbf{b}$ and $T(\mathbf{b}+\mathbf{c}) = \mathbf{a}$.

a. What is $T^{-1}(\mathbf{a})$?
b. What is $T(\mathbf{a}+\mathbf{b}+\mathbf{c})$ in its simplest form?
c. What is $T(\mathbf{a})$?

Question 9. — Write $\begin{pmatrix} 1 & 1 \\ 1 & 100 \end{pmatrix}$ as a product of elementary matrices.

Question 10. — Suppose that A and B are invertible $n \times n$ matrices. Find the inverse of

$$\begin{pmatrix} A & I \\ B^{-1}A & 0 \end{pmatrix}.$$

Question 11. — Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 7 \end{pmatrix}$.

a. Evaluate the determinant of A.
b. Use Cramer's rule to solve $A\mathbf{x} = 2\mathbf{e}_3$ for x_4 only.

Question 12. — Let $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and $V = \{\mathbf{x} \in \mathbb{R}^2 \colon \mathbf{x} \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{v}\}$.

a. Find a basis of V.
b. Draw a picture of V.

Question 13. — For $n \times n$ matrices A and B, let

$$V = \{X \in M_{n\times n} \colon AXB = BXA\}.$$

a. Show that V is a subspace of $M_{n\times n}$.
b. Show that if A is invertible then $A^{-1} \in V$.

Question 14. — Let $S = \{1 + 2x - x^2, 1 + 3x^2, 4x + ax^2\}$.

a. For which values of a is S linearly dependent?
b. What is the dimension of $\mathrm{Span}\, S$ if $a = 0$?

Question 15. — Let

$$\ell_1 = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\} \qquad \text{and} \qquad \ell_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

a. Find the distance between ℓ_1 and ℓ_2.
b. Find an equation of the line through the origin which intersects ℓ_1 at a right angle.

Question 16. — Let T be the triangle with vertices $A(1, 1, 1)$, $B(2, 2, 2)$ and $C(3, -1, 2)$.

a. Find a normal equation of the plane containing T.
b. Find the area of T.
c. Find the cosine of the angle at vertex A of T.
d. Find the point on side AC which is 2 units from A.

Question 17. — Let $\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

a. Compute $\mathbf{e}_1 \times \mathbf{w}$, $\mathbf{e}_2 \times \mathbf{w}$ and $\mathbf{e}_3 \times \mathbf{w}$.
b. Express $\|\mathbf{e}_1 \times \mathbf{w}\|^2 + \|\mathbf{e}_2 \times \mathbf{w}\|^2 + \|\mathbf{e}_2 \times \mathbf{w}\|^2$ in terms of $\|\mathbf{w}\|$.

Question 18. — Fill in the correct numerical value for each of the following statements.

a. If A is a 7×4 matrix of rank 3 and $A\mathbf{x} = b$ is consistent, then the rank of the 7×5 matrix $\begin{pmatrix} A & \mathbf{b} \end{pmatrix}$ is _____.
b. If $\left\{\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ a \\ b \end{pmatrix}\right\}$ is linearly dependent, then $a =$ _____ and $b =$ _____.
c. If $\det(A) = 5$ and $\det(2A) = 40$ then $\det(3A) =$ _____.
d. If A is a 6×6 matrix such that $\mathrm{Row}(A)$, $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$ all have the same dimension, then the rank of A is _____.

Solution to Question 1. — a. By inspecting R, the solution of $A\mathbf{x} = 0$ is

$$\operatorname{Span}\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\0\\0\\1\end{pmatrix}\right\}.$$

b. The dimension of $\operatorname{Nul}(A)$ is 3.

c. The pivot columns of A,

$$\mathbf{a}_1 = \begin{pmatrix}0\\2\\1\end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix}2\\0\\2\end{pmatrix}$$

form a basis of the column space of A.

d. The vector in question belongs to the column space of A if, and only if, it is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_3$:

$$s\begin{pmatrix}0\\2\\1\end{pmatrix} + t\begin{pmatrix}2\\0\\2\end{pmatrix} = \begin{pmatrix}6\\a\\0\end{pmatrix}.$$

Plainly this can only happen if $t = 3$ (look at row 1) and $s = -6$ (look at row 3), which gives $a = -12$ (look at row 2).

e. Only $\{\mathbf{a}_2, \mathbf{a}_4\}$ is a basis of $\operatorname{Col}(A)$ as it is the only list of two linearly independent vectors (and the rank of A is 2).

Solution to Question 2. — Reducing the augmented matrix of the system gives

$$(A\ \ \mathbf{b}) \sim \begin{pmatrix}1 & 3 & 1 & 0\\0 & -1 & 2 & 3\\0 & -2 & a-1 & 6\\0 & 1 & -2 & b\end{pmatrix} \sim \begin{pmatrix}1 & 3 & 1 & 0\\0 & -1 & 2 & 3\\0 & 0 & a-5 & 0\\0 & 0 & 0 & b+3\end{pmatrix}$$

a. The system is inconsistent if $b \neq -3$, regardless of the value of a.

b. The system has a unique solution if $b = -3$ and $a \neq 5$.

c. The system has infinitely many solutions if $b = -3$ and $a = 5$.

d. The system is inconsistent unless $b = -3$, in which case the third entry of the solution must be zero unless $a - 5$. For these values of a and b the given vector is indeed a solution of the equation.

Solution to Question 3. — The coefficients of the required polynomial form the solution of $A\mathbf{x} = \mathbf{b}$, where

$$(A\ \ \mathbf{b}) = \begin{pmatrix}1 & 1 & 1 & 5\\1 & 2 & 4 & 1\\1 & -1 & 1 & 1\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1 & 5\\0 & 1 & 3 & -4\\0 & -2 & 0 & -4\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -2 & 9\\0 & 1 & 3 & -4\\0 & 0 & 6 & -12\end{pmatrix}$$

$$\sim \begin{pmatrix}1 & 0 & 0 & 5\\0 & 1 & 0 & 2\\0 & 0 & 1 & -2\end{pmatrix}.$$

Thus, the required polynomial is $5 + 2x - 2x^2$.

Solution to Question 4. — a. The vector $\mathbf{u} = \mathbf{e}_1 - 2\mathbf{e}_2$ is a direction vector to the line defined by $y = -2x$ (so it is parallel to this line).

b. The vector $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2$ is orthogonal to $\mathbf{u}$, so it is orthogonal to the line defined by $y = -2x$.

c. Writing $\mathbf{x}$ for the vector in question, the orthogonality of $\mathbf{u}, \mathbf{v}$ gives

$$\begin{pmatrix}5\\15\end{pmatrix} = \mathbf{x} = \frac{\mathbf{u}^T\mathbf{x}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} + \frac{\mathbf{v}^T\mathbf{x}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = -5\begin{pmatrix}1\\-2\end{pmatrix} + 5\begin{pmatrix}2\\1\end{pmatrix}.$$

Solution to Question 5. — a. The matrix A is symmetric (*i.e.*, $A^T = A$) if, and only if, $a = 1$ (b can be any real number).

b. Since

$$A^2 = \begin{pmatrix}3 & a\\1 & b\end{pmatrix}\begin{pmatrix}3 & a\\1 & b\end{pmatrix} = \begin{pmatrix}a+9 & a(b+3)\\b+3 & a+b^2\end{pmatrix},$$

$A^2 = A$ if, and only if, $a = -6$ and $b = -2$.

c. From part b, $A^{-1} = A$ if, and only if, $a = -8$ and $b = -3$.

d. Since $\det(A) = 3b - a$, A is invertible if, and only if, $a \neq 3b$.

Solution to Question 6. — Since A has mutually orthogonal columns of length 2, and $A^T = A$, it follows that $A^2 = A^TA = 4I_4$ and $A^{-1} = \frac{1}{4}A$.

Solution to Question 7. — a. The standard matrices of S and R are

$$B = [S] = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix} \qquad \text{and} \qquad S = [R] = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},$$

so the standard matrix of $R \circ S$ is

$$[R \circ S] = AB = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}.$$

b. The linear transformation $R \circ S$ maps $\mathbf{e}_1 \rightsquigarrow -\mathbf{e}_2$ and $\mathbf{e}_2 \rightsquigarrow \mathbf{e}_1$, which effects a rotation about the origin through $\frac{3}{2}\pi$ (radians).

Solution to Question 8. — a. Since $T(\mathbf{b}+\mathbf{c}) = \mathbf{a}$, $T^{-1}(\mathbf{a}) = \mathbf{b}+\mathbf{c}$.

b. Since $\mathbf{a}+\mathbf{b}+\mathbf{c} = \frac{1}{2}\big((\mathbf{b}+\mathbf{c}) + (\mathbf{a}+\mathbf{c}) + (\mathbf{a}+\mathbf{b})\big)$, it follows that

$$T(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

c. Since $T(\mathbf{b}+\mathbf{c}) = \mathbf{a}$, the result of part b gives

$$T(\mathbf{a}) = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}+\mathbf{c}) - \mathbf{a} = \tfrac{1}{2}(-\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Solution to Question 9. — The elementary row operations $R_2 \leftarrow R_2 - R_1$, $R_2 \leftarrow \frac{1}{99}R_2$, $R_1 \leftarrow R_1 - R_2$ yield

$$\begin{pmatrix}1 & 1\\1 & 100\end{pmatrix} \sim \begin{pmatrix}1 & 1\\0 & 99\end{pmatrix} \sim \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.$$

Thus,

$$\begin{pmatrix}1 & 1\\1 & 100\end{pmatrix} = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 99\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}.$$

Solution to Question 10. — Since

$$\begin{pmatrix}A & I\\B^{-1}A & 0\end{pmatrix}\begin{pmatrix}0 & A^{-1}B\\I & -B\end{pmatrix} = \begin{pmatrix}I & 0\\0 & I\end{pmatrix},$$

it follows that

$$\begin{pmatrix}A & I\\B^{-1}A & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & A^{-1}B\\I & -B\end{pmatrix}.$$

Solution to Question 11. — a. By a direct computation,

$$\det(A) = \begin{vmatrix}1 & 0 & 0 & 0\\1 & 2 & 2 & 2\\1 & 2 & 4 & 4\\1 & 2 & 4 & 6\end{vmatrix} = 8\begin{vmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 1 & 2\end{vmatrix} = 8.$$

b. Since

$$\det(A_4(2\mathbf{e}_3)) = 2\begin{vmatrix}1 & 0 & 0 & 0\\1 & 2 & 2 & 0\\1 & 2 & 4 & 1\\1 & 2 & 4 & 0\end{vmatrix} = 8\begin{vmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 1 & 0\end{vmatrix} = -8,$$

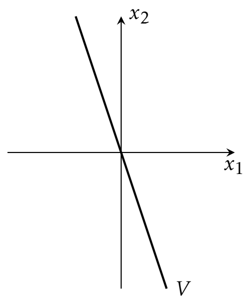
it follows that $x_4 = -1$.

Solution to Question 12. — a. Note that V is the set of all vectors $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{w}^T\mathbf{x} = 0$, where

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = \begin{pmatrix}3\\1\end{pmatrix}, \qquad \text{so} \qquad \left\{\begin{pmatrix}-1\\3\end{pmatrix}\right\}$$

is a basis for V.

b. Below is a picture of V.



Solution to Question 13. — a. If $X, Y \in V$ and $\alpha, \beta \in \mathbb{R}$, then $AXB = BXA$ and $AYB = BYA$, so

$$A(\alpha X + \beta Y)B = \alpha AXB + \beta AYB = \alpha BXA + \beta BYA = B(\alpha X + \beta Y)A;$$

thus $\alpha X + \beta Y \in V$. Since V is non-empty (the $n \times n$ zero matrix belongs to V), it follows that V is a subspace of $M_{n\times n}$.

b. If A is invertible, then $AA^{-1}B = B = BA^{-1}A$, and thus $A^{-1} \in V$.

Solution to Question 14. — a. The reduction

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 4 \\ -1 & 3 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 0 & a+8 \end{pmatrix}$$

implies that the dependence equation

$$\alpha(1 + 2x - x^2) + \beta(1 + 3x^2) + \gamma(4x + ax^2) = 0$$

has non-trivial solutions if, and only if, $a = -8$.

b. By part a S is linearly independent unless $a = -8$, so the dimension of $\operatorname{Span} S$ is 3 if $a = 0$.

Solution to Question 15. — a. Let

$$\mathbf{u} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \text{so} \quad \mathbf{u} \times \mathbf{v} = \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix}.$$

The distance between ℓ_1 and ℓ_2 is

$$\frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{(-4)^2 + 1^2 + 7^2}}{\sqrt{2^2 + 1^2 + 1^2}} = \sqrt{11}.$$

b. The line which contains the origin and intersects ℓ_1 at a right angle is $\operatorname{Span}\{\operatorname{perp}_{\mathbf{v}} \mathbf{w}\}$, where $\mathbf{v}$ is from part a and

$$\mathbf{w} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \quad \text{so} \quad \operatorname{perp}_{\mathbf{v}} \mathbf{w} = \mathbf{w} - \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} - \frac{7}{3}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}.$$

Solution to Question 16. — Let

$$\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \text{so} \quad \mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}.$$

a. The plane of T is defined by $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\,\overrightarrow{OA}$, or $3x_1 + x_2 - 4x_3 = 0$.
b. The area of T is $\frac{1}{2}\|\mathbf{n}\| = \frac{1}{2}\sqrt{26}$.
c. The cosine of the angle in T at A is

$$\frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{3\sqrt{3}} = \frac{1}{9}\sqrt{3}.$$

d. The point on side AC which is 2 units from A is

$$\overrightarrow{OA} + \frac{2}{\|\mathbf{v}\|}\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}.$$

Solution to Question 17. — a. Direct computations gives

$$\mathbf{e}_1 \times \mathbf{w} = \begin{pmatrix} 0 \\ -c \\ b \end{pmatrix}, \quad \mathbf{e}_2 \times \mathbf{w} = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} \quad \text{and} \quad \mathbf{e}_3 \times \mathbf{w} = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}.$$

b. By part a, $\|\mathbf{e}_1 \times \mathbf{w}\|^2 + \|\mathbf{e}_2 \times \mathbf{w}\|^2 + \|\mathbf{e}_3 \times \mathbf{w}\|^2 = 2(a^2 + b^2 + c^2) = 2\|\mathbf{w}\|^2$.

Solution to Question 18. — a. If A is a 7×4 matrix of rank 3 and $A\mathbf{x} = b$ is consistent, then the rank of the 7×5 matrix $\begin{pmatrix} A & \mathbf{b} \end{pmatrix}$ is $\underline{3}$.

b. If $\left\{\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ a \\ b \end{pmatrix}\right\}$ is linearly dependent, then $a = \underline{\frac{4}{3}}$ and $b = \underline{-\frac{14}{3}}$.

c. If $\det(A) = 5$ and $\det(2A) = 40$ then $\det(3A) = \underline{135}$.

d. If A is a 6×6 matrix such that $\operatorname{Row}(A)$, $\operatorname{Col}(A)$ and $\operatorname{Nul}(A)$ all have the same dimension, then the rank of A is $\underline{3}$.