1. Given that $A = \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

a. Solve $A\mathbf{x} = \mathbf{b}$.

b. Write $\mathbf{b}$ as a linear combination of the columns of A.

c. What is rank(A)?

d. What is the dimension of Nul(A^T)?

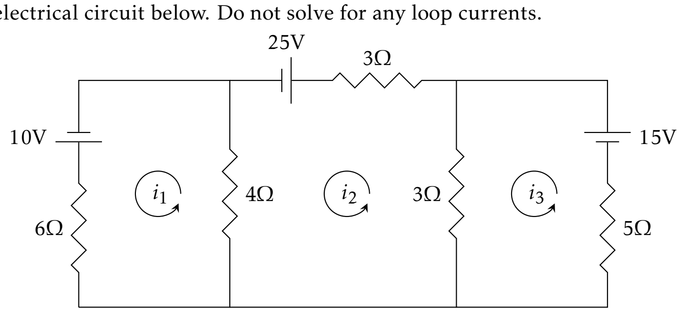
e. Is $-\mathbf{e}_1 + \mathbf{e}_2$ in Nul(A^T)? Justify.

2. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ k^2-5 \end{pmatrix}$ and $\mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix}$.

a. For what values of k is $\mathbf{v}_4$ in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. For what values of k is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent?

3. Write a matrix equation whose solution gives the loop currents in the electrical circuit below. Do not solve for any loop currents.



4. Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ -1 & -1 & -3 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 3 & 12 \\ 1 & 8 & 5 \end{pmatrix}$.

a. Find an LU factorization of A.

b. Write L as a product of elementary matrices.

6. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the horizontal expansion by a factor of 2, and let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the vertical shear which transforms $(1,0)$ into $(1,2)$.

a. Write the standard matrix of S. b. Write the standard matrix of T.

c. Write the standard matrix of the composite $S \circ T$.

d. Sketch the triangle $\mathscr{T}$ in $\mathbb{R}^2$ with vertices $(-2,0)$, $(2,0)$ and $(0,4)$.

e. Sketch the image $(S \circ T)(\mathscr{T})$. f. Compute area of $(S \circ T)(\mathscr{T})$.

7. Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}.$$

The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\mathbf{x}) = A\mathbf{x}$, and the plane $\mathscr{P}$ has parametric vector equation $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$.

a. Is the image $T(\mathscr{P})$ a plane, a line or a point? Justify.

b. Is T injective? Justify.

8. Let A, B and C be 3×3 matrices. If $\det(A) = 10$, $\det(B) = -2$ and C is non-invertible, evaluate the following determinants.

a. $\det(3B^2A^{-1})$ b. $\det(C^TA + C^TB)$ c. $\det((3A)^{-1}B^2)$

9. Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{vmatrix} = 3$, compute the following determinants.

a. $\begin{vmatrix} 2 & 2 & 2 \\ a & b & c \\ d-3 & e-3 & f-3 \end{vmatrix}$ b. $\begin{vmatrix} 0 & 0 & 5 & 10 \\ a & d & 2 & 5 \\ b & e & 2 & 5 \\ c & f & 2 & 5 \end{vmatrix}$

10. Let $M = \begin{pmatrix} I & A \\ A & I \end{pmatrix}$ and $N = \begin{pmatrix} I & 0 \\ -A & I \end{pmatrix}$,

where A is an $n \times n$ matrix, I is the $n \times n$ identity matrix and $A^2 = I$.

a. Compute and simplify MN. b. Is M invertible? Justify.

11. Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors in $\mathbb{R}^n$ which are orthogonal to each other. Compute $(2\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - 5\mathbf{v})$ and $\|\mathbf{u} + 4\mathbf{v}\|$.

12. Find the point between $P(6,-2,5)$ and $Q(10,2,7)$ whose distance from P is 2 units.

13. Find the values of h and k for which the line $\mathbf{x} = \begin{pmatrix} h \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ k \\ 3 \end{pmatrix}$ lies in the plane $x - 2y + z = 5$.

14. You are given the following points: $A(1,2,3)$, $B(2,2,4)$, $C(-5,3,1)$.

a. Give a parametric vector equation of the line containing A and B.

b. Find the point on the line from part a which is closest to the point C.

c. Find the area of the triangle whose vertices are the points A, B and C.

15. Let $\mathscr{H} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x^2 + y^2 = z^2 \right\}$.

a. List two matrices in $\mathscr{H}$, neither of which is a scalar multiple of the other.

b. Is $\mathscr{H}$ closed under scalar multiplication? Justify.

c. Is $\mathscr{H}$ closed under addition? Justify.

16. Find a basis the subspace $\mathscr{W} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = -2\mathbf{x}\}$ of $\mathbb{R}^3$, where

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}.$$

17. Complete each statement with **must**, **might** or **cannot**, as appropriate.

a. If $T\colon \mathbb{R}^6 \to \mathbb{R}^8$ is a linear transformation, then T ____ be onto.

b. If A is row equivalent to B then Col(A) ____ be equal to Col(B).

c. If $\mathbf{u}$, $\mathbf{v}$ are non-zero vectors and $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \mathbf{u}$, then $\mathbf{u}$ ____ be parallel to $\mathbf{v}$.

d. If $\mathbf{u}$, $\mathbf{v}$ are non-zero vectors in $\mathbb{R}^3$, then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$ ____ be equal to 0.

e. If A is a 3×3 matrix, B is a 4×4 matrix and rank(A) = rank(B), then $\det(A)$ ____ be equal to zero and $\det(B)$ ____ be equal to zero.

18. Give a 3×5 matrix $B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{pmatrix}$ which satisfies all four of the following conditions.

- B is in reduced echelon form.
- $\mathbf{b}_1$, $\mathbf{b}_3$ are pivot columns of B.
- $\{\mathbf{b}_2, \mathbf{b}_4\}$ is a basis of Col(B).
- $\{\mathbf{b}_2, \mathbf{b}_5\}$ is not a basis of Col(B).

1. Writing $A = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5\end{pmatrix}$, observe that $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_5$ are linearly independent, $\mathbf{a}_1 = 2\mathbf{a}_2 + \mathbf{a}_3$, $\mathbf{a}_4 = \mathbf{a}_2 + 2\mathbf{a}_3$ and $\mathbf{b} = 11\mathbf{a}_2 + 7\mathbf{a}_3 - 5\mathbf{a}_5$. The rank of A is three. Since $\dim(\mathrm{Nul}(A^T)) = 0$, $-\mathbf{e}_1 + \mathbf{e}_2$ certainly does not belong to $\mathrm{Nul}(A^T)$. Finally, the solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{p} + \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{p} = \begin{pmatrix}0\\11\\7\\0\\-5\end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix}1\\-2\\-1\\0\\0\end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix}0\\-1\\-2\\1\\0\end{pmatrix}.$$

2. Reducing $V = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{pmatrix}$ to an echelon form gives

$$\begin{pmatrix}1 & 1 & -1 & 2\\ 1 & 2 & 1 & 3\\ 1 & 1 & k^2-5 & k\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1 & 2\\ 0 & 1 & -3 & -1\\ 0 & 0 & k^2-4 & k-2\end{pmatrix}.$$

So $\mathbf{v}_4 \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if, and only if, $k \neq -2$; *i.e.*, $\mathbf{v}_4$ is not a pivot column of V. Also, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent if, and only if, $k \neq \pm 2$; *i.e.*, $\mathbf{v}_3$ is a pivot column of $\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{pmatrix}$.

3. Applying Kirchhoff's and Ohm's laws gives the equation $R\mathbf{i} = \mathbf{V}$, where

$$R = \begin{pmatrix}10 & -4 & 0\\ -4 & 10 & -3\\ 0 & -3 & 8\end{pmatrix}, \qquad \mathbf{i} = \begin{pmatrix}i_1\\ i_2\\ i_3\end{pmatrix} \qquad \text{and} \qquad \mathbf{V} = \begin{pmatrix}10\\ -25\\ 15\end{pmatrix}.$$

4. Reducing $\begin{pmatrix}A & I_3\end{pmatrix}$ gives

$$\begin{pmatrix}A & I_3\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 0 & 1 & -2 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & 5 & -3 & -2\\ 0 & 1 & 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -2 & 1 & 0\end{pmatrix},$$

so

$$A^{-1} = \begin{pmatrix}5 & -3 & -2\\ 1 & 0 & 1\\ -2 & 1 & 0\end{pmatrix}.$$

5. An LU factorization of A is

$$\begin{pmatrix}2 & 4 & 6\\ 3 & 3 & 12\\ 1 & 8 & 5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ \frac{3}{2} & 1 & 0\\ \frac{1}{2} & -2 & 1\end{pmatrix}\begin{pmatrix}2 & 4 & 6\\ 0 & -3 & 3\\ 0 & 0 & 8\end{pmatrix},$$

via the rough work

$$\boxed{\begin{matrix}-3 & 3\\ 6 & 2\end{matrix}} \quad \rightsquigarrow \quad \boxed{8},$$

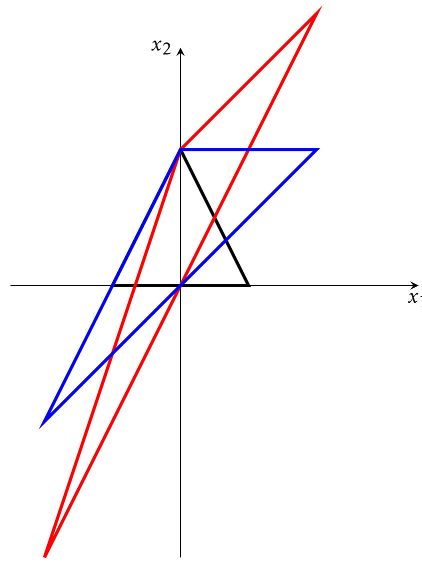
and

$$\begin{pmatrix}1 & 0 & 0\\ \frac{3}{2} & 1 & 0\\ \frac{1}{2} & -2 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ \frac{3}{2} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -2 & 1\end{pmatrix}.$$

6. The standard matrices A of S, B of T, AB of $S \circ T$ and BA of $T \circ S$ are

$$A = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \quad B = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}, \quad AB = \begin{pmatrix}2 & 0\\ 4 & 1\end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix}2 & 0\\ 2 & 1\end{pmatrix}.$$

In the figure below, the boundary of $\mathscr{T}$ is drawn in black, the boundary of $(S \circ T)(\mathscr{T})$ is drawn in red, and the boundary of $(T \circ S)(\mathscr{T})$ is drawn in blue.



$$(S \circ T)\begin{pmatrix}\pm 2\\ 0\end{pmatrix} = \begin{pmatrix}\pm 4\\ \pm 8\end{pmatrix}$$

$$(S \circ T)\begin{pmatrix}0\\ 4\end{pmatrix} = \begin{pmatrix}0\\ 4\end{pmatrix}$$

$$(T \circ S)\begin{pmatrix}\pm 2\\ 0\end{pmatrix} = \begin{pmatrix}\pm 4\\ \pm 4\end{pmatrix}$$

$$(T \circ S)\begin{pmatrix}0\\ 4\end{pmatrix} = \begin{pmatrix}0\\ 4\end{pmatrix}$$

The area of $(T \circ S)(\mathscr{T})$ is 16, which may be seen in many ways; *e.g.*, $\frac{1}{2} \cdot 4 \cdot 8$ (*i.e.*, half the product of its base and its height), or $2 \cdot \frac{1}{2} \cdot 4 \cdot 4$ (*i.e.*, the product of the absolute value of $\det(BA)$ and the area of $\mathscr{T}$).

7. Since $A\begin{pmatrix}\mathbf{p} & \mathbf{u} & \mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & 2(\mathbf{e}_1 - \mathbf{e}_3) & \mathbf{e}_1 - \mathbf{e}_3\end{pmatrix}$, it follows that the image of $\mathscr{P}$ under the action of T is the line $\mathrm{Span}\{\mathbf{e}_1 - \mathbf{e}_3\}$. Certainly T is not injective, since $\mathbf{p}$ is a non-zero element of its kernel.

8. a. $\det(3B^2A^{-1}) = 3^3(-2)^2 10^{-1} = \frac{54}{5}$, as the determinant is multilinear, invariant under transposition and preserves products.

b. Since C is singular and the determinant is invariant under transposition $\det(C^T) = 0$, so the determinant of $C^TA + C^TB = C^T(A+B)$ is zero, since the determinant preserves products.

c. $\det((3A)^{-1}B^2) = (3^3 \cdot 10)^{-1}(-2)^2 = \frac{2}{135}$, as the determinant is multilinear, invariant under transposition and preserves products.

9. a. The alternating multilinearity of the determinant implies that

$$\begin{vmatrix}2 & 2 & 2\\ a & b & c\\ d-3 & e-3 & f-3\end{vmatrix} = (-1)^2 2\begin{vmatrix}a & b & c\\ d & e & f\\ 1 & 1 & 1\end{vmatrix} = 6.$$

b. The multilinearity of the determinant, Laplace expansion along the first row, and invariance under transposition, gives

$$\begin{vmatrix}0 & 0 & 5 & 10\\ a & d & 2 & 5\\ b & e & 2 & 5\\ c & f & 2 & 5\end{vmatrix} = 5 \cdot 5\begin{vmatrix}a & b & c\\ d & e & f\\ 1 & 1 & 1\end{vmatrix} - 5 \cdot 2 \cdot 2\begin{vmatrix}a & b & c\\ d & e & f\\ 1 & 1 & 1\end{vmatrix} = 15.$$

10. If $\mathbf{a}_1$ denotes the first column of A, then

$$MN = \begin{pmatrix}I & A\\ A & I\end{pmatrix}\begin{pmatrix}I & 0\\ -A & I\end{pmatrix} = \begin{pmatrix}I - A^2 & A\\ A - A & I\end{pmatrix} = \begin{pmatrix}0 & A\\ 0 & I\end{pmatrix}, \quad \text{and so} \quad \begin{pmatrix}\mathbf{e}_1\\ -\mathbf{a}_1\end{pmatrix}$$

is a non-zero vector in the null space of M. Hence, M is not invertible.

11. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are mutually orthogonal unit vectors, then the symmetry and bilinearity of the scalar product implies that

$$(2\mathbf{u} + \mathbf{v})^T(\mathbf{u} - 5\mathbf{v}) = 2\mathbf{u}^T\mathbf{u} - 9\mathbf{u}^T\mathbf{v} - 5\mathbf{v}^T\mathbf{v} = -3$$

and

$$(\mathbf{u} + 4\mathbf{v})^T(\mathbf{u} + 4\mathbf{v}) = \mathbf{u}^T\mathbf{u} + 8\mathbf{u}^T\mathbf{v} + 16\mathbf{v}^T\mathbf{v} = 17,$$

so $\|\mathbf{u} + 4\mathbf{v}\| = \sqrt{17}$.

12. The length of $\overrightarrow{PQ} = (4,4,2) = 2(2,2,1)$ is 6, so the point $P + \frac{1}{3}\overrightarrow{PQ}$, *i.e.*, $(6,-2,5) + \frac{2}{3}(2,2,1) = \frac{1}{3}(22,-2,17)$ is between P and Q and 2 units from P.

13. The line lies in the plane if, and only if,

$$\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} h & 1 \\ 2 & k \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} h-5 \\ 4-2k \end{pmatrix}, \qquad \text{is equal to} \qquad \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

or equivalently, $h = 10$ and $k = 2$.

14. Write $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ for the position vectors, respectively, of A, B, C; also

$$\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix} -6 \\ 1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{u} \times \mathbf{v} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}.$$

a. The line containing A and B is $\mathbf{a} + \mathrm{Span}\{\mathbf{u}\}$.

b. The point on the line containing A and B which is closest to C is

$$\mathbf{a} + \mathrm{proj}_{\mathbf{u}}\,\mathbf{v} = \mathbf{a} + \frac{\mathbf{u}^T\mathbf{v}}{\mathbf{u}^T\mathbf{u}}\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}.$$

c. The area of triangle ABC is $\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{18} = \frac{3}{2}\sqrt{2}$.

15. a. Taking $x = 1$, $y = 0$ and $z = \pm 1$ gives two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which belong to $\mathscr{H}$, neither of which is a scalar multiple of each other.

b. If $\alpha \in \mathbb{R}$ and

$$A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \mathscr{H}, \qquad \text{then} \qquad \alpha A = \begin{pmatrix} \alpha x & \alpha y \\ 0 & \alpha z \end{pmatrix},$$

and $(\alpha x)^2 + (\alpha y)^2 = \alpha^2(x^2 + y^2) = \alpha^2 z^2 = (\alpha z)^2$, since $x^2 + y^2 = z^2$, so $\alpha A \in \mathscr{H}$. Therefore, $\mathscr{H}$ is closed under scalar multiplication.

c. The sum of the matrices given in part a is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which does not belong $\mathscr{H}$. Therefore, $\mathscr{H}$ is not closed under addition.

16. The subspace $\mathscr{W}$ is the null space of

$$A + 2I_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{for which a basis is} \qquad \left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\},$$

since the first and second columns of this matrix are linearly independent, and its third column is the sum of its first two columns.

17. a. If $T\colon \mathbb{R}^6 \to \mathbb{R}^8$ is a linear transformation, then T **cannot** be onto.

b. If A is row equivalent to B then $\mathrm{Col}(A)$ **might** be equal to $\mathrm{Col}(B)$.

c. If $\mathbf{u}$, $\mathbf{v}$ are non-zero vectors and $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u} = \mathbf{u}$, then $\mathbf{u}$ **must** be parallel to $\mathbf{v}$.

d. If $\mathbf{u}$, $\mathbf{v}$ are non-zero vectors in $\mathbb{R}^3$, then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$ **must** be equal to 0.

e. If A is a 3×3 matrix, B is a 4×4 matrix and $\mathrm{rank}(A) = \mathrm{rank}(B)$, then $\det(A)$ **might** be equal to zero and $\det(B)$ **must** be equal to zero.

18. Such a matrix is obtained by taking $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}_5 = \mathbf{e}_1$ and $\mathbf{b}_3 = \mathbf{b}_4 = \mathbf{e}_3$, which gives $B = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix}$.