Question 1. — Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & a & 2 \\ 1 & 2 & a^2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$. Find all values of a for which $A\mathbf{x} = \mathbf{b}$ has: a. a unique solution; b. infinitely many solutions; c. no solution.

Question 2. — Find the polynomial $p(x) = x^4 + ax^2 + bx^2 + cx + d$ such that $p(1) = -1$, $p(-1) = 1$, $p'(1) = -7$ and $p'(-1) = -3$.

Question 3. — Find bases of $\mathrm{Col}(A)$, $\mathrm{Row}(A)$ and $\mathrm{Nul}(A^T)$, where

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 1 \\ 4 & 0 & 8 & 4 & 4 \\ 2 & 0 & 3 & 2 & 1 \end{pmatrix}.$$

Question 4. — Show that if $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$, then $3\mathbf{u} - 4\mathbf{v}$ is a solution to $A\mathbf{x} = 3\mathbf{b}$.

Question 5. — Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical expansion by a factor of 2 and let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation about the origin by $\frac{1}{2}\pi$ (radians).

a. Find the standard matrix of $S \circ T$.

b. Let $\mathscr{R}$ be the triangular region with vertices $(-1,-1)$, $(4,0)$ and $(3,2)$. Make two sketches, one of $\mathscr{R}$ and the other of the image $(S \circ T)(\mathscr{R})$.

Question 6. — a. Given that B is invertible, find the inverse of

$$\begin{pmatrix} B & 0 \\ C & 2I \end{pmatrix}.$$

b. Use part a to compute the inverse of

$$\begin{pmatrix} 2 & 4 & 0 & 0 & 0 \\ -1 & -5 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ 2 & -3 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

c. What is the inverse of

$$\begin{pmatrix} 2 & -1 & -1 & 2 & 2 \\ 4 & -5 & 1 & -3 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}?$$

Question 7. — Find A^{-1} and express A as a product of elementary matrices, where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 2 \\ 1 & 3 & 12 \end{pmatrix}.$$

Question 8. — Suppose that A and B are invertible $n \times n$ matrices and B is symmetric. Is the matrix $AB^{-1}A^T - B$ is also symmetric? Justify your answer.

Question 9. — Solve the the equation $(3XB)^{-1} + A = X^{-1}$ for the matrix X.

Question 10. — Compute the determinant

$$\begin{vmatrix} 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & -3 \end{vmatrix}.$$

Question 11. — Given that A, B, C are $2{\times}2$ matrices, $\det(A) = 3$, $\det(B) = -2$ and $\det(C) = 0$, find: a. $\det((3AB^2)^{-1})$; b. $\det(AC + BC)$; c. $\det(A^{-1} + \mathrm{adj}(A))$.

Question 12. — Let $\mathscr{H} = \{A \in M_{2\times 2} : \det(A) = 0\}$.

a. Find two matrices in $\mathscr{H}$, neither of which is a scalar multiple of the other.

b. Is $\mathscr{H}$ closed under addition?

c. Is $\mathscr{H}$ closed under scalar multiplication?

d. Is $\mathscr{H}$ a subspace of $M_{2\times 2}$?

Question 13. — Find a basis of $\mathscr{S} = \{A \in M_{2\times 2} : A\mathbf{v} = \mathbf{0}\}$, where $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Question 14. — Let W be the set of all polynomials $\mathbf{p}(x) \in \mathbb{P}_3[x]$ such that $\mathbf{p}(1) = 0$ and $\mathbf{p}'(-1) = 0$. Find a basis for W.

Question 15. — Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \ell = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix}\right\}.$$

a. Find a normal equation of the plane ϖ spanned by $\mathbf{u}$ and $\mathbf{v}$.

b. Show that the line ℓ is entirely contained in the plane ϖ.

Question 16. — Given $Q(1,6,0)$, $\mathscr{P}\colon x - 2y + z = 3$ and $\ell\colon \mathbf{x} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

a. Find an equation of the line through the origin and Q.

b. Find the cosine of the angle between $\mathscr{P}$ and the yz-plane.

c. Find the distance between Q and ℓ.

d. Find the point on $\mathscr{P}$ which is closest to Q.

e. Find a normal equation of the plane which is perpendicular to $\mathscr{P}$ and contains ℓ.

Question 17. — Let $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ k \\ k^2 \end{pmatrix}$.

a. Find all values of k for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b. Find a unit vector which is orthogonal to $\mathbf{u}$.

Question 18. — Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

a. Simplify $\mathbf{u} \cdot \big((\mathbf{v} - \mathbf{w}) \times (\mathbf{w} - \mathbf{u})\big)$.

b. True or false: the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ has the same volume as the parallelepiped determined by $\mathbf{u}$, $\mathbf{v} - \mathbf{u}$, $\mathbf{w} - \mathbf{u}$.

Question 19. — Show that if $\{\mathbf{a}, \mathbf{b}\}$ is linearly independent, then $\{\mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}\}$ is linearly independent.

Question 20. — Complete each sentence with **must**, **might** or **cannot**.

a. If A is a product of elementary matrices, then $\det(A)$ ______ equal zero.

b. Two lines in $\mathbb{R}^3$ that are both perpendicular to a third line ______ be parallel.

c. If the matrix AB is invertible, then A ______ be invertible.

d. If $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ is a linear transformation, then the kernel of T ______ be a plane.

Solution to Question 1. — Since

$$\begin{pmatrix} A & \mathbf{b} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & a-4 & 0 & 0 \\ 0 & 0 & a^2-1 & -1 \end{pmatrix},$$

it follows that $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution if, and only if, $a \neq \pm 1, 4$;
b. infinitely many solutions if, and ony if, $a = 4$;
c. no solution if, and only if, $a = \pm 1$.

Solution to Question 2. — The conditions $p(1) = -1$ and $p(-1) = 1$ are equivalent to $a+b+c+d = -2$ and $-a+b-c+d = 0$, or $a+c = b+d = -1$. Next, since $p'(x) = 4x^3 + 3ax^2 + 2bx + c$, the conditions $p'(1) = -7$ and $p'(-1) = -3$ are equivalent to $3a + 2b + c = -11$ and $3a - 2b + c = 1$, which gives $4b = -12$, or $b = -3$ and $2a = -4$, or $a = -2$. It follows that $c = 1$ and $d = 2$, so $p(x) = x^4 - 2x^3 - 3x^2 + x + 2$.

Solution to Question 3. — Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{pmatrix}$ and $A^T = \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{pmatrix}$. Now $\mathbf{a}_1$, $\mathbf{a}_3$ are linearly independent, $\mathbf{a}_2 = \mathbf{0}$, $\mathbf{a}_4 = \mathbf{a}_1$ and $\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_3$; also, $\mathbf{r}_1$, $\mathbf{r}_3$ are linearly independent, and $\mathbf{r}_2 = 4\mathbf{r}_1$. So $\{\mathbf{a}_1, \mathbf{a}_3\}$ is a basis of $\mathrm{Col}(A)$, $\{\mathbf{r}_1, \mathbf{r}_3\}$ is a basis of $\mathrm{Row}(A)$, and $\{4\mathbf{e}_1 - \mathbf{e}_2\}$ is a basis of $\mathrm{Nul}(A^T)$.

Solution to Question 4. — Since $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{0}$, it follows that

$$A(3\mathbf{u} - 4\mathbf{v}) = 3A\mathbf{u} - 4A\mathbf{v} = 3\mathbf{b}.$$

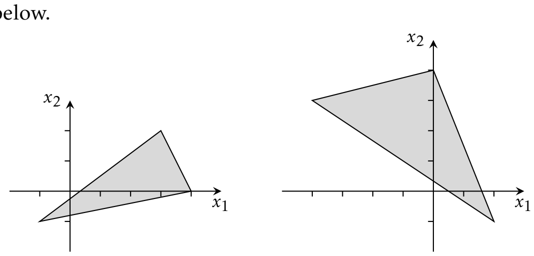
Solution to Question 5. — Since $S \circ T$ maps

$$\mathbf{e}_1 \rightsquigarrow \mathbf{e}_1 \rightsquigarrow \mathbf{e}_2 \qquad \text{and} \qquad \mathbf{e}_2 \rightsquigarrow 2\mathbf{e}_2 \rightsquigarrow -2\mathbf{e}_1,$$

its standard matrix is

$$\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 4 & 3 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -4 \\ -1 & 4 & 3 \end{pmatrix},$$

are the images of the vertices of the triangle. The triangle and its image are shown below.



Solution to Question 6. — a. A direct calculation gives

$$\frac{1}{2}\begin{pmatrix} 2B^{-1} & 0 \\ -CB^{-1} & I \end{pmatrix}\begin{pmatrix} B & 0 \\ C & 2I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \text{so} \quad \begin{pmatrix} B & 0 \\ C & 2I \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 2B^{-1} & 0 \\ -CB^{-1} & I \end{pmatrix},$$

since the given matrix is square.

b. The matrix in question has the given form, where

$$B = \begin{pmatrix} 2 & 4 \\ -1 & -5 \end{pmatrix}, \qquad \text{so} \qquad 2B^{-1} = \frac{1}{3}\begin{pmatrix} 5 & 4 \\ -1 & -2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} -1 & 1 \\ 2 & -3 \\ 2 & 1 \end{pmatrix}, \qquad \text{so} \quad -CB^{-1} = \frac{1}{6}\begin{pmatrix} -1 & 1 \\ 2 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -5 & -4 \\ 1 & 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 6 & 6 \\ -13 & -14 \\ -9 & -6 \end{pmatrix},$$

giving the (non-zero and non-identity) blocks of (twice) the inverse.

c. The inverse of the matrix is the transpose of the inverse from part b.

Solution to Question 7. — Reducing A to I_3 by the sequence $R_1 \leftrightarrow R_3$, $R_1 \leftarrow R_1 - 12R_3$, $R_2 \leftarrow R_2 - 2R_3$, $R_1 \leftarrow R_1 + 6R_2$, $R_2 \leftarrow -2R_2$ of elementary row operations, and applying the same sequence of row operations to I_3 gives

$$A \sim \begin{pmatrix} 1 & 3 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } I_3 \sim \begin{pmatrix} -12 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -24 & 6 & 1 \\ 4 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A^{-1}.$$

The matrix A is the product of the inverses of the corresponding elementary matrices, taken in the order of application in the row reduction of A to I_3:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution to Question 8. — Since $(AB^{-1}A^T)^T = A(B^T)^{-1}A^T = AB^{-1}A^T$, B is symmetric, and and transposition is linear, the matrix in question is symmetric.

Solution to Question 9. — The given equation is equivalent to

$$A = X^{-1} - (3B)^{-1}X^{-1} = (I - (3B)^{-1})X^{-1},$$

and thus $X = A^{-1}(I - (3B)^{-1}) = A^{-1} - (3BA)^{-1}$.

Solution to Question 10. — A direct calculation gives

$$\det(A) = \begin{vmatrix} 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & -1 \\ 1 & 2 & 2 \\ -1 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 3.$$

Solution to Question 11. — a. $\det\big((3AB^2)^{-1}\big) = \frac{1}{3^2 \cdot 3 \cdot (-2)^2} = \frac{1}{108}$.

b. $\det(AC + BC) = \det(A + B)\det(C) = 0$, since C is singular.

c. $\det\big(A^{-1} + \mathrm{adj}(A)\big) = \det(4A^{-1}) = \frac{4^2}{3} = \frac{16}{3}$, since $\mathrm{adj}(A) = 3A^{-1}$.

Solution to Question 12. — a. $X = \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \end{pmatrix}$ and $Y = \begin{pmatrix} \mathbf{0} & \mathbf{e}_1 \end{pmatrix}$ are two linearly indendent matrices in $\mathscr{H}$.

b. Since $X, Y \in \mathscr{H}$ from part a, but $X + Y = I_2 \notin \mathscr{H}$, it follows that $\mathscr{H}$ is not closed under addition.

c. If $A \in \mathscr{H}$ and α is any scalar, then $\det(\alpha A) = \alpha^2 \det(A) = 0$, so $\alpha A \in \mathscr{H}$. Thus, $\mathscr{H}$ is closed under scalar multiplication.

d. Since $\mathscr{H}$ is not closed under addition, it is not a subspace of $M_{2\times 2}$.

Solution to Question 13. — A matrix $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix} \in \mathscr{S}$ if, and only if, $2\mathbf{a}_1 - 3\mathbf{a}_2 = \mathbf{0}$, so

$$\left\{ \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \right\}$$

is a basis of $\mathscr{S}$.

Solution to Question 14. — If $\mathbf{p} \in W$ then $\mathbf{p}(x) = \alpha + (\beta x + \gamma)(x+1)^2$ for some scalars α, β, γ. If, in addition, $\mathbf{p}(1) = 0$ then $\alpha + 4\beta + 4\gamma = 0$. Thus, $\{-4 + x(x+1)^2, -4 + (x+1)^2\}$ is a basis of W.

Solution to Question 15. — a. The plane $\varpi = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{n}^T\mathbf{x} = 0\}$, where

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}.$$

b. If

$$\mathbf{p} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix},$$

then $\mathbf{n}^T\begin{pmatrix} \mathbf{p} & \mathbf{w} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$, so the line $\mathbf{p} + \mathrm{Span}\{\mathbf{w}\}$ is contained in the plane $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$. Alternatively, one could observe that $\mathbf{p} = 2\mathbf{u}$ and $\mathbf{w} = -\mathbf{u} + 5\mathbf{v}$.

Solution to Question 16. — Let

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \mathbf{p} - \mathbf{q} = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}.$$

a. The line through the origin and $\mathbf{q}$ is $\mathrm{Span}\{\mathbf{q}\}$.

b. The cosine of the angle between $\mathscr{P}$ and the yz-plane is

$$\frac{|\mathbf{n}^T\mathbf{e}_1|}{\|\mathbf{n}\|\,\|\mathbf{e}_1\|} = \frac{1}{6}\sqrt{6}.$$

c. The distance from $\mathbf{q}$ to $\ell = \mathbf{p} + \mathrm{Span}\{\mathbf{v}\}$ is $\|\mathrm{perp}_{\mathbf{v}}(\mathbf{p} - \mathbf{q})\| = \sqrt{3}$, since

$$\mathrm{perp}_{\mathbf{v}}(\mathbf{p} - \mathbf{q}) = \mathbf{p} - \mathbf{q} - \frac{\mathbf{v}^T(\mathbf{p} - \mathbf{q})}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

d. The point on $\mathscr{P}$ which is closest to $\mathbf{q}$ is

$$\mathbf{q}+\frac{3-\mathbf{n}^T\mathbf{q}}{\mathbf{n}^T\mathbf{n}}\mathbf{n}=\begin{pmatrix}1\\6\\0\end{pmatrix}+\frac{7}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}10\\4\\7\end{pmatrix}.$$

e. The plane in question is parallel to both $\mathbf{n}$ and $\mathbf{v}$, so it is orthogonal to

$$\mathbf{m}=\tfrac{1}{4}\mathbf{n}\times\mathbf{v}=\tfrac{1}{4}\begin{pmatrix}0\\4\\8\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix},$$

and is thus defined by $\mathbf{m}^T\mathbf{x}=\mathbf{m}^T\mathbf{p}$, or $y+2z=7$

Solution to Question 17. — a. Since $\mathbf{u}^T\mathbf{v}=-6-k+k^2=(k+2)(k-3)$, it follows that $\mathbf{u}$ is orthogonal to $\mathbf{v}$ if, and only if, $k=-2,3$.

b. The unit vector $\frac{1}{2}\sqrt{2}\,(\mathbf{e}_2+\mathbf{e}_3)$ is orthogonal to $\mathbf{u}$.

Solution to Question 18. — a. The expression

$$\mathbf{u}\cdot\big((\mathbf{v}-\mathbf{w})\times(\mathbf{w}-\mathbf{u})\big)=\det\big(\mathbf{u}\quad\mathbf{v}-\mathbf{u}\quad\mathbf{w}-\mathbf{u}\big)=\det\big(\mathbf{u}\quad\mathbf{v}\quad\mathbf{w}\big).$$

b. By part a, the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is equal to the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}-\mathbf{u}$ and $\mathbf{w}-\mathbf{u}$.

Solution to Question 19. — If $\alpha(\mathbf{a}-\mathbf{b})+\beta(\mathbf{a}+\mathbf{b})=\mathbf{0}$, or equivalently, $(\alpha+\beta)\mathbf{a}+(\beta-\alpha)\mathbf{b}=\mathbf{0}$, then $\alpha+\beta=0$ and $\beta-\alpha=0$, and thus $\alpha=\beta=0$, since $\mathbf{a},\mathbf{b}$ are linearly independent. Therefore, $\mathbf{a}-\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$ are linearly independent.

Solution to Question 20. — a. If A is a product of elementary matrices, then $\det(A)$ **cannot** equal zero.

b. Two lines in $\mathbb{R}^3$ which are both perpendicular to a third line **might** be parallel.

c. If the matrix AB is invertible, then A **might** be invertible.

d. The kernel of a linear transformation $T\colon\mathbb{R}^4\to\mathbb{R}^2$ **might** be a plane.