Question 1. — Consider the following systems of linear equations.

$$\begin{aligned} x + y - z &= 0 \\ x - y + 2z &= 0 \\ 3x + y &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + y - z &= a \\ x - y + 2z &= b \\ 3x + y &= c \end{aligned}$$

a. Find the general solution of the first system.
b. If $x = 1$, $y = 1$, $z = 1$ is a solution of the second system, find a, b and c, and give the general solution of the second system.

Question 2. — Show that for any square matrix A and any positive integer n, every vector in $\mathrm{Nul}(A)$ belongs to $\mathrm{Nul}(A^n)$.

Question 3. — Find a quadratic polynomial whose graph contains $(-1,6)$, $(1,24)$ and $(2,48)$.

Question 4. — Consider the matrix $A = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5\end{pmatrix}$ and its reduced echelon form R below.

$$A = \begin{pmatrix} 4 & 5 & -12 & 3 & 8 \\ 3 & 1 & 2 & 5 & 17 \\ -2 & -1 & 0 & 2 & -5 \\ 5 & 2 & 2 & -1 & 18 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using only the columns of A, give two distinct bases for $\mathrm{Col}(A)$.

Question 5. — Consider the matrix equation $A^{-1}B = (C - 2A)^{-1}$.

a. Solve the given equation for A.
b. If $B = \begin{pmatrix} 4 & 1 \\ -3 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 3 \\ 5 & -3 \end{pmatrix}$ in the given equation, evaluate A.

Question 6. — Show that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$, then $\|\mathbf{u}\| = \|\mathbf{v}\|$.

Question 7. — You are given 4×4 matrices A and B such that $\det(A^2B) = 20$ and $\det(AB^2) = 50$. Evaluate $\det(A)$, $\det(B)$, $\det(A^{-1})$ and $\det(3B^T)$.

Question 8. — Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = n \neq 0 \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3b + 2c \\ 3e + 2f \\ 3h + 2i \end{pmatrix},$$

use Cramer's rule to solve for x_3 only.

Question 9. — Find all values of k for which the linear system

$$\begin{aligned} 2kx + (k+1)y &= 2 \\ x + y + z &= 0 \\ -kx + (1-2k)y &= -1 \end{aligned}$$

has: a. no solution; b. a unique solution; c. infinitely many solutions.

Question 10. — Show that for any matrix A, $A^T(4A)$ is symmetric.

Question 11. — Let R be the reduced echelon form of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ -5 & 3 & -7 \end{pmatrix}.$$

Find R, and express A as a product of R and some elementary matrices.

Question 12. — Let $T\colon \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & x \\ 2 & 1 & y \\ 0 & 3 & z \end{pmatrix}$$

a. Evaluate $T(\mathbf{e}_1 + 2\mathbf{e}_2 + 5\mathbf{e}_3)$. b. Find the standard matrix of T.
c. Find a basis for $\ker(T)$.

Question 13. — Find an LU-factorization of $A = \begin{pmatrix} 2 & -6 & -2 & 4 \\ -1 & 0 & 3 & 2 \\ -1 & 15 & 7 & 10 \end{pmatrix}$.

Question 14. — Let A, B, C be matrices with A and C invertible.

a. Compute the inverse of $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$.
b. Use the result of part a to compute the inverse of

$$\begin{pmatrix} 1/2 & 0 & 0 & 1 & 1 \\ 0 & 1/2 & 0 & 1 & 1 \\ 0 & 0 & 1/2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

Question 15. — You are given the points $A(3,1,1)$, $B(2,1,3)$ and $C(1,0,3)$.

a. Find the distance from B to the line AC.
b. Find the point on the line AC which is closest to B.
c. Find the cosine of the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
d. Find the area of the triangle ABC.

Question 16. — Let $\ell = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$ and $\ell' = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}\right\}$.
Find a normal equation of the plane which contains ℓ and is parallel to ℓ'.

Question 17. — Let $V = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : y^2 = 4x^2\right\}$.

a. Is V closed under vector addition? Justify.
b. Is V closed under scalar multiplication? Justify.

Question 18. — Let $p(x) = 2 + x - x^2$, $q(x) = 3 + 2x + 2x^2$, $r(x) = 3 + 4x + 16x^2$.

a. Show that $r(x) \in \mathrm{Span}\{p(x), q(x)\}$.
b. Is $\{p(x), q(x), r(x)\}$ a basis for $\mathbb{P}_2[x]$? Justify your answer.

Question 19. — Compute the standard matrix of $S \circ T$, where $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ rotates points about the origin by $\frac{1}{3}\pi$ (radians) and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ projects points (orthogonally) onto the x_1-axis.

Question 20. — Let $\mathscr{H} = \left\{A \in M_{2\times2} : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A = A \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\right\}$.

a. Given that $\mathscr{H}$ is a subspace of $M_{2\times2}$, find a basis of $\mathscr{H}$.
b. What is the dimension of $\mathscr{H}$?
c. Can

$$\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$$

be written as a linear combination of the basis from part a? Justify.

Question 21. — Complete each sentence with **must**, **might** or **cannot**.

a. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^3$, then $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ ________ be equal to $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})$.

b. If A is a square matrix and $A^4 - 2A^2 + A = I$ then A ________ be invertible.

c. If A and B are $n\times n$ matrices such that $AB = B$, then A ________ be an identity matrix.

d. If $A\mathbf{x} = \mathbf{b}$ has two distinct solutions then the columns of A ________ be linearly dependent.

e. If ℓ_1 is a line in $\mathbb{R}^2$ which does not contain the origin, ℓ_2 is a line in $\mathbb{R}^2$ which contains the origin and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(\ell_1) = \ell_2$, then T ________ be injective (one-to-one).

f. If A is an $m\times n$ matrix and $\mathrm{Nul}(A) = \mathbb{R}^n$ then A ________ be the $m\times n$ zero matrix.

Solution to Question 1. — a. Reducing the coefficient matrix of the system gives

$$\begin{pmatrix}1&1&-1\\1&-1&2\\3&1&0\end{pmatrix}\sim\begin{pmatrix}1&1&-1\\0&-2&3\\0&-2&3\end{pmatrix}\sim\begin{pmatrix}1&1&\frac{1}{2}\\0&1&-\frac{3}{2}\\0&0&0\end{pmatrix},\qquad\text{so if}\qquad \mathbf{u}=\begin{pmatrix}1\\-3\\-2\end{pmatrix},$$

then the solution of the system is $\operatorname{Span}\{\mathbf{u}\}$.

b. Since the first and second systems have the same coefficient matrix, the solution of the second system is $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\operatorname{Span}\{\mathbf{u}\}$.

Solution to Question 2. — Suppose that $n\ge 1$ and $\mathbf{x}\in\operatorname{Nul}(A)$. If $n=1$ then $A\mathbf{x}=\mathbf{0}$ by definition, and if $n>1$ then $A^n\mathbf{x}=A^{n-1}A\mathbf{x}=A^{n-1}\mathbf{0}=\mathbf{0}$, since $n-1$ is a positive integer. Thus, $\mathbf{x}\in\operatorname{Nul}(A^n)$. in any case.

Solution to Question 3. — By inspection, the polynomial is given by

$$\begin{aligned}p(x)&=\frac{6(x-1)(x-2)}{(-1-1)(-1-2)}+\frac{24(x+1)(x-2)}{(1+1)(1-2)}+\frac{48(x+1)(x-1)}{(2+1)(2-1)}\\&=(x-1)(x-2)-12(x+1)(x-2)+16(x+1)(x-1).\end{aligned}$$

Solution to Question 4. — Two such bases are given by $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_4\}$ and $\{\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}$; the latter since $\mathbf{a}_2,\mathbf{a}_3$ are linearly independent, $\mathbf{a}_1=2\mathbf{a}_2+\frac{1}{2}\mathbf{a}_3$ and $\mathbf{a}_4\notin\operatorname{Span}\{\mathbf{a}_2,\mathbf{a}_3\}=\operatorname{Span}\{\mathbf{a}_1,\mathbf{a}_2\}$.

Solution to Question 5. — a. The equation is question is equivalent to $B^{-1}=(C-2A)A^{-1}=CA^{-1}-2I$, or $A=(B^{-1}+2I)^{-1}C$.

b. From part a,

$$A=\left(\begin{pmatrix}1&1\\-3&-4\end{pmatrix}+\begin{pmatrix}2&0\\0&2\end{pmatrix}\right)^{-1}C=\tfrac{1}{3}\begin{pmatrix}2&1\\-3&-3\end{pmatrix}\begin{pmatrix}2&3\\5&-3\end{pmatrix}=\begin{pmatrix}3&1\\-7&0\end{pmatrix}$$

Solution to Question 6. — If $0=(\mathbf{u}+\mathbf{v})^T(\mathbf{u}-\mathbf{v})=\mathbf{u}^T\mathbf{u}-\mathbf{v}^T\mathbf{v}=\|\mathbf{u}\|^2-\|\mathbf{v}\|^2$, then $\|\mathbf{u}\|=\|\mathbf{v}\|$ (since the length of a vector is non-negative).

Solution to Question 7. — Let $x=\det(A)$ and $y=\det(B)$. It is given that $x^2y=20$ and $xy^2=50$, so $x^3y^3=1000$, or $xy=10$. Thus, $\det(A)=x=2$, $\det(B)=y=5$, $\det(A^{-1})=\frac{1}{2}$ and $\det(3B^T)=3^4\cdot 5=405$.

Solution to Question 8. — By inspection it is plain that the solution is $3\mathbf{e}_3+2\mathbf{e}_4$ so $x_3=3$. Cramer's rule would give

$$x_3=\frac{\begin{vmatrix}1&0&0&0\\0&a&3b+2c&c\\0&d&3e+2f&f\\0&g&3h+2i&i\end{vmatrix}}{\begin{vmatrix}1&0&0&0\\0&a&b&c\\0&d&e&f\\0&g&h&i\end{vmatrix}}=\frac{\begin{vmatrix}a&3b&c\\d&3e&f\\g&3h&i\end{vmatrix}}{\begin{vmatrix}a&b&c\\d&e&f\\g&h&i\end{vmatrix}}=3.$$

Solution to Question 9. — Reducing the augmented matrix of the system with the second equation written first gives

$$\begin{pmatrix}1&1&1&0\\0&1-k&-2k&2\\0&1-k&k&-1\end{pmatrix}\sim\begin{pmatrix}1&1&1&0\\0&1-k&-2k&2\\0&0&3k&-3\end{pmatrix}.$$

Thus, the system has no solution if $k=0$, infinitely many solutions if $k=1$ and a unique solution if $k\neq 0,1$.

Solution to Question 10. — Since $A^T(4A)=4A^TA$ and $(A^TA)^T=A^TA$, it follows that $A^T(4A)$ is symmetric.

Solution to Question 11. — Since

$$A\sim\begin{pmatrix}1&0&2\\0&0&0\\0&3&3\end{pmatrix}\sim\begin{pmatrix}1&0&2\\0&1&1\\0&0&0\end{pmatrix}=R,$$

via $R_2\leftarrow R_2-2R_1$, $R_3\leftarrow R_3+5R_1$, $R_2\leftrightarrow R_3$ and $R_2\leftarrow\frac{1}{3}R_2$, it follows that

$$A=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-5&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}R.$$

Solution to Question 12. — Note that

$$T(\mathbf{x})=\left(\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\3\end{pmatrix}\right)^T\mathbf{x}=\begin{pmatrix}6\\-3\\1\end{pmatrix}^T\mathbf{x},$$

and thus

a. $T(\mathbf{e}_1+2\mathbf{e}_2+5\mathbf{e}_3)=5$,

b. the standard marix of T is $\begin{pmatrix}6&-3&1\end{pmatrix}$ and

c. $\left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\3\end{pmatrix}\right\}$ is a basis of the kernel of T.

Solution to Question 13. — An LU-factorization of A is

$$\begin{pmatrix}2&-6&-2&4\\-1&0&3&2\\-1&15&7&10\end{pmatrix}=\begin{pmatrix}1&0&0\\-\frac{1}{2}&1&0\\-\frac{1}{2}&-4&1\end{pmatrix}\begin{pmatrix}2&-6&-2&4\\0&-3&2&4\\0&0&14&28\end{pmatrix},$$

via the rough work

$$\begin{matrix}-3&2&4\\12&6&12\end{matrix}\quad\rightsquigarrow\quad 14\quad 28.$$

Solution to Question 14. — a. The inverse of the matrix in question is the unique solution of

$$\begin{pmatrix}A&B\\0&C\end{pmatrix}\begin{pmatrix}X&Y\\0&Z\end{pmatrix}=\begin{pmatrix}I&0\\0&I\end{pmatrix},\qquad\text{or}\qquad\begin{pmatrix}AX&AY+BZ\\0&CZ\end{pmatrix}=\begin{pmatrix}I&0\\0&I\end{pmatrix}.$$

Thus, $X=A^{-1}$, $Z=C^{-1}$ and $Y=-A^{-1}BC^{-1}$.

b. In this case $X=2I_3$,

$$Z=\begin{pmatrix}-1&2\\2&-3\end{pmatrix}\quad\text{and}\quad Y=-2I_3\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix}\begin{pmatrix}-1&2\\2&-3\end{pmatrix}=\begin{pmatrix}-2&2\\-2&2\\-2&2\end{pmatrix}.$$

Therefore,

$$\begin{pmatrix}1/2&0&0&1&1\\0&1/2&0&1&1\\0&0&1/2&1&1\\0&0&0&3&2\\0&0&0&2&1\end{pmatrix}^{-1}=\begin{pmatrix}2&0&0&-2&2\\0&2&0&-2&2\\0&0&2&-2&2\\0&0&0&-1&2\\0&0&0&2&-3\end{pmatrix}.$$

Solution to Question 15. — Let

$$\mathbf{u}=\overrightarrow{AB}=\begin{pmatrix}-1\\0\\2\end{pmatrix},\qquad\mathbf{v}=\overrightarrow{AC}=\begin{pmatrix}-2\\-1\\2\end{pmatrix}\qquad\text{and}\qquad\mathbf{w}=\mathbf{u}\times\mathbf{v}=\begin{pmatrix}2\\-2\\1\end{pmatrix}.$$

a. The distance between B and the line AC is $\|\mathbf{w}\|/\|\mathbf{v}\|=1$.

b. The point on the line AC which is closest to B is

$$\overrightarrow{OA}+\operatorname{proj}_{\mathbf{v}}\mathbf{u}=\begin{pmatrix}3\\1\\1\end{pmatrix}+\tfrac{2}{3}\begin{pmatrix}-2\\-1\\2\end{pmatrix}=\tfrac{1}{3}\begin{pmatrix}5\\1\\7\end{pmatrix}.$$

c. The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{6}{3\sqrt{5}}=\tfrac{2}{5}\sqrt{5}.$$

d. The area of $\triangle ABC$ is $\frac{1}{2}\|\mathbf{w}\|=\frac{3}{2}$.

Solution to Question 16. — The plane in question is orthogonal to

$$\mathbf{n}=-\begin{pmatrix}1\\0\\2\end{pmatrix}\times\begin{pmatrix}-1\\1\\4\end{pmatrix}=\begin{pmatrix}2\\6\\-1\end{pmatrix}\qquad\text{and contains}\qquad\mathbf{p}=\begin{pmatrix}1\\2\\2\end{pmatrix};$$

so the plane is defined by $\mathbf{n}^T\mathbf{x}=\mathbf{n}^T\mathbf{p}$, or $2x_1+6x_2-x_3=12$.

Solution to Question 17. — a. Since

$$\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix}\in V\qquad\text{but}\qquad\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}\notin V,$$

it follows that V is not closed under addition.

b. If $y^2 = 4x^2$ and $\alpha \in \mathbb{R}$, then $(\alpha y)^2 = \alpha^2 y^2 = \alpha^2 4x^2 = 4(\alpha x)^2$; so if

$$\begin{pmatrix} x \\ y \end{pmatrix} \in V \qquad \text{then} \qquad \alpha \begin{pmatrix} x \\ y \end{pmatrix} \in V.$$

Therefore, V is closed under scalar multiplication. that if

c.

Solution to Question 18. — a. Since

$$\begin{pmatrix} 2 & 3 & 3 \\ 1 & 2 & 4 \\ -1 & 2 & 16 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 3 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & \frac{7}{2} & \frac{35}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix},$$

it follows that $r(x) = -6p(x) + 5q(x) \in \operatorname{Span}\{p(x), q(x)\}$.

b. Since $\{p(x), q(x), r(x)\}$ is linearly dependent (by part a), it cannot be a basis of $\mathbb{P}_2[x]$ (or of any linear space).

Solution to Question 19. — The transformation $S \circ T$ maps

$$\mathbf{e}_1 \rightsquigarrow \tfrac{1}{2}\mathbf{e}_1 + \tfrac{1}{2}\sqrt{3}\mathbf{e}_2 \rightsquigarrow \tfrac{1}{2}\mathbf{e}_1 \quad \text{and} \quad \mathbf{e}_2 \rightsquigarrow -\tfrac{1}{2}\sqrt{3}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 \rightsquigarrow -\tfrac{1}{2}\sqrt{3}\mathbf{e}_1,$$

so its standard matrix is

$$\tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ 0 & 0 \end{pmatrix}.$$

Solution to Question 20. — a. Writing E_{ij} for the 2×2 matrix with 1 in row i and column j and 0 elsewhere, the coordinates of $A \in \mathscr{H}$ relative to $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ will belong to the null space of

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $\mathscr{H}$.

b. The dimension of $\mathscr{H}$ is 2.

c. The given matrix does not belong to $\mathscr{H}$ (since the sum of the entries in its second column is not zero), so it is not a linear combination of the basis vectors from part a.

Solution to Question 21. — a. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbb{R}^3$, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ **cannot** be equal to $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$.

b. If A is a square matrix and $A^4 - 2A^2 + A = I$ then A **must** be invertible.

c. If A and B are $n \times n$ matrices such that $AB = B$, then A **might** be an identity matrix.

d. If $A\mathbf{x} = \mathbf{b}$ has two distinct solutions then the columns of A **must** be linearly dependent.

e. If ℓ_1 is a line in $\mathbb{R}^2$ which does not contain the origin, ℓ_2 is a line in $\mathbb{R}^2$ which contains the origin and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T(\ell_1) = \ell_2$, then T **cannot** be injective (one-to-one).

f. If A is an $m \times n$ matrix and $\operatorname{Nul}(A) = \mathbb{R}^n$ then A **must** be the $m \times n$ zero matrix.