Question 1. — Given $A = \begin{pmatrix} 2 & -4 & 2 & 2 \\ 3 & -7 & 2 & 2 \\ 4 & -7 & 5 & 3 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ -9 \\ -1 \end{pmatrix}$.

a. Find the general solution of $A\mathbf{x} = \mathbf{b}$.
b. Find a specific solution of $A\mathbf{x} = \mathbf{b}$ such that $x_1 = x_2$.
c. Write the third column of A as a linear combination of the first two columns of A, or else explain why this is not possible.
d. True or false: There is a vector $\mathbf{c} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{c}$ has no solution.
e. True or false: The last three columns of A form an invertible matrix.

Question 2. — The graph of $y = ax^2 + bx + c$ contains the points $(3, 27)$ and $(2, -6)$. The tangent line to the graph where $x = 2$ has slope 12. Write (but do not solve) an equation (or linear system) whose solution gives a, b and c.

Question 3. — Given $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 2 & 3 & -1 \\ 0 & -1 & 8 \end{pmatrix}$.

a. Find A^{-1}.
b. Given that $\det(A) = \frac{1}{2}$, find $\operatorname{adj}(A)$.

Question 4. — Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 10$, find $\begin{vmatrix} 3g+a & 3h+b & 2 & 3i+c \\ d+2a & e+2b & 3 & f+2c \\ a & b & 4 & c \\ 0 & 0 & 5 & 0 \end{vmatrix}$.

Question 5. — Given that the matrices A and B are invertible, solve $A^{-1}B(X+I)^{-1}A = 2A$ for X.

Question 6. — You are given that A is invertible and $B = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}$.

a. Find B^{-1}.
b. Find B^4.
c. If $A^5 = I$ and $A \neq I$, find the smallest positive integer m such that $B^m = I$.

Question 7. — You are given $A = \begin{pmatrix} 3 & 6 & 3 & 21 \\ -2 & -4 & -2 & -14 \\ -1 & -2 & 2 & 2 \end{pmatrix} \sim R = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

a. Find a basis for $\operatorname{Col}(A)$.
b. Find a basis for $\operatorname{Row}(A)$.
c. Find a basis for $\operatorname{Nul}(A)$.
d. State the dimension of $\operatorname{Nul}(A^T)$.
e. Select the answer which correctly completes the following sentence.
The column space of A is ________. (empty / a point / a line / a plane / $\mathbb{R}^3$ / $\mathbb{R}^4$).

Question 8. — Let A be an $m \times n$ matrix with $n > m$.

a. Explain why $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions.
b. What is the size of A^TA?
c. Explain why A^TA cannot be invertible.

Question 9. — Let $\mathscr{H} = \{A \in M_{2\times 2} : A^2 = A^T + A\}$.

a. For which c is $\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \in \mathscr{H}$?
b. Show that $\mathscr{H}$ is not a subspace of $M_{2\times 2}$.

Question 10. — Find a basis of $\mathscr{V} = \{p \in \mathbb{P}_3 : p(1) = p(2)\}$.

Question 11. — Given each of the following matrices, indicate whether its columns are linearly dependent or linearly independent.

a. A 4×5 matrix with a pivot in each row.
b. The product of two elementary matrices.
c. The standard matrix of an injective linear transformation.

Question 12. — Given that A, B, C are $n \times n$ matrices,

$$\det(A) = 2, \qquad \det(AB) = 6, \qquad \det(2A) = 32 \qquad \text{and} \qquad \dim \operatorname{Nul}(AC) = 1,$$

find each of the following. a. $\det(A^5A^T)$ b. $\det(B^{-1})$ c. n d. $\det(-A)$ e. $\det(BC)$ f. $\operatorname{rank}(AB)$

Question 13. — Find an LU factorization of $\begin{pmatrix} 3 & 4 & 1 \\ 9 & 11 & 5 \\ -6 & 13 & 8 \end{pmatrix}$.

Question 14. — You are given the lines $\ell_1 = \mathbf{p}_1 + \operatorname{Span}\{\mathbf{v}_1\}$ and $\ell_2 = \mathbf{p}_2 + \operatorname{Span}\{\mathbf{v}_2\}$, where

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ -4 \\ -2 \end{pmatrix}, \qquad \mathbf{p}_2 = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

a. Find the point of intersection of ℓ_1 and ℓ_2.
b. Find the cosine of the angle ϑ between ℓ_1 and ℓ_2.
c. Is the angle ϑ between 0 and $\frac{1}{3}\pi$?

Question 15. — Find the point on the line $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}\right\}$ which is nearest to $(-9, 15, -1)$.

Question 16. — Find the area of the triangle with vertices $A(4, 2, 1)$, $B(3, 1, 5)$ and $C(2, 3, 6)$.

Question 17. — Find an equation of the line which contains the origin and is parallel to the planes defined by $x + y + z = 1$ and $2x - y + z = 3$.

Question 18. — Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the y-axis and let $S: \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical shear which maps $\mathbf{e}_1 + 2\mathbf{e}_2$ to $\mathbf{e}_1 - 5\mathbf{e}_2$.

a. Find the standard matrix of R.
b. Find the standard matrix of S.
c. Find the standard matrix of $S^{-1} \circ R$.

Question 19. — Let Q be a solid object with volume 7, and let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & k & 0 \\ 2 & 5 & 2 \end{pmatrix}.$$

Find all values of k, if any, so that the volume of $T(Q)$ is 35.

Question 20. — Complete each sentence with **must**, **might** or **cannot**.

a. Let A be a square matrix. If $A\mathbf{x} = A\mathbf{y}$ for distinct $\mathbf{x}, \mathbf{y}$, then A ________ be invertible.
b. Let T be a linear transformation with standard matrix A. The kernel of T ________ equal the null space of A and the range of T ________ equal the column space of A.
c. If $\mathbf{w}$ is in $\operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$ then $\mathbf{w}$ ________ be in $\{\mathbf{u}, \mathbf{v}\}$.
d. Let A be a non-zero, non-invertible 3×3 matrix. If the column space of A does not form a line, then the null space of A ________ form a line.

Question 21. — Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be such that $\|\mathbf{u}\| = 3$, $\mathbf{v}$ is a unit vector, and $\mathbf{u} + 2\mathbf{v}$ is orthogonal to $\mathbf{u} + 3\mathbf{v}$. Find $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{u} + \mathbf{v}\|$.

Question 22. — Prove that if $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and $\mathbf{w} \notin \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

Solution to Question 1. — a. Reducing $\begin{pmatrix} A & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{b} \end{pmatrix}$ gives

$$\begin{pmatrix} A & \mathbf{b} \end{pmatrix} \sim \begin{pmatrix} 2 & -4 & 2 & 2 & 0 \\ 0 & -1 & -1 & -1 & -9 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & -4 & 2 & 2 & 0 \\ 0 & -1 & -1 & -1 & -9 \\ 0 & 0 & 0 & -2 & -10 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & -4 & 2 & 0 & -10 \\ 0 & -1 & -1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 0 & 3 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}.$$

Thus, the general solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{p} + \operatorname{Span}\{\mathbf{u}\}$, where

$$\mathbf{p} = \begin{pmatrix} 3 \\ 4 \\ 0 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

b. Solving $3 - 3x = 4 - x$ gives $x = -\frac{1}{2}$, so

$$\mathbf{p} - \tfrac{1}{2}\mathbf{u} = \tfrac{1}{2}\begin{pmatrix} 9 \\ 9 \\ -1 \\ 10 \end{pmatrix}$$

is a solution of $A\mathbf{x} = \mathbf{b}$ whose first and second entries are equal.

c. From the reduced echelon form of A, it follows that $\mathbf{a}_3 = 3\mathbf{a}_1 + \mathbf{a}_2$.

d. Since A has three pivot columns, its columns span $\mathbb{R}^3$; therefore, the statement is false.

e. The matrix whose columns are (in order) the last three columns of A is row equivalent to

$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim I_3,$$

so the statement in question is true.

Solution to Question 2. — If $y = c + bx + ax^2$ then $\frac{dy}{dx} = b + 2ax$, so the unique solution of the equation

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} -6 \\ 27 \\ 12 \end{pmatrix}$$

yields the coefficients of 1, x and x^2 in y.

Solution to Question 3. — a. The matrix of cofactors of A is

$$C = \begin{pmatrix} 23 & -16 & -2 \\ -\frac{11}{2} & 4 & \frac{1}{2} \\ -5 & \frac{7}{2} & \frac{1}{2} \end{pmatrix},$$

and the determinant of A is $\frac{1}{2}$ (this can be seen by adding 8 times the second column to the third column and then expanding; but it is also given), so

$$A^{-1} = 2\operatorname{adj}(A) = 2\begin{pmatrix} 23 & -\frac{11}{2} & -5 \\ -16 & 4 & \frac{7}{2} \\ -2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 46 & -11 & -10 \\ -32 & 8 & 7 \\ -4 & 1 & 1 \end{pmatrix}.$$

b. The adjoint of A is displayed in the solution to part a.

Solution to Question 4. — Expanding along the bottom row, and then applying the multilinear and alternating character of the determinant, gives

$$\begin{vmatrix} 3g+a & 3h+b & 2 & 3i+c \\ d+2a & e+2b & 3 & f+2c \\ a & b & 4 & c \\ 0 & 0 & 5 & 0 \end{vmatrix} = -5\begin{vmatrix} 3g & 3h & 3i \\ d & e & f \\ a & b & c \end{vmatrix} = 15\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 150.$$

Solution to Question 5. — The equation is equivalent to $(X+I)^{-1} = 2B^{-1}A$, or $X = (2B^{-1}A)^{-1} - I$.

Solution to Question 6. — a. The equation

$$\begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = I,$$

is equivalent to the equations $AY = I$, $W = 0$, $AZ = 0$, $X = I$. Thus, since B is square, it follows that

$$B^{-1} = \begin{pmatrix} 0 & I \\ A^{-1} & 0 \end{pmatrix}.$$

b. Since

$$B^2 = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \text{it follows that} \quad B^4 = \begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix}.$$

c. From parts a and b (and induction) it follows that

$$B^{2k} = \begin{pmatrix} A^k & 0 \\ 0 & A^k \end{pmatrix}, \quad \text{and} \quad B^{2k+1} = \begin{pmatrix} A^k & 0 \\ 0 & A^k \end{pmatrix}\begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^{k+1} \\ A^k & 0 \end{pmatrix},$$

for any integer k. Therefore, the smallest positive integer m for which $B^m = I$ is 10.

Solution to Question 7. — Write $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{pmatrix}$ and $R^T = \begin{pmatrix} \mathbf{r} & \mathbf{r}' & \mathbf{0} \end{pmatrix}$.

a. A basis for $\operatorname{Col}(A)$ is $\{\mathbf{a}_1, \mathbf{a}_3\}$.

b. A basis for $\operatorname{Row}(A)$ is $\{\mathbf{r}, \mathbf{r}'\}$.

c. A basis for $\operatorname{Nul}(A)$ is

$$\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

d. The dimension of $\operatorname{Nul}(A^T)$ is $3 - \operatorname{rank}(A) = 1$.

e. Since $\operatorname{rank}(A) = 2$, the column space of A is **a plane**.

Solution to Question 8. — a. The dimension of $\operatorname{Nul}(A)$ is $n - \operatorname{rank}(A) \geqslant n - m > 0$.

b. The matrix A^TA is an $n \times n$ matrix.

c. A^TA must be singular because $\dim \operatorname{Nul}(A^TA) \geqslant \dim \operatorname{Nul}(A) > 0$.

Solution to Question 9. — a. Since

$$\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1+c^2 & 2c \\ 2c & 1+c^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} + \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2c \\ 2c & 2 \end{pmatrix},$$

it follows that the given matrix belongs to $\mathscr{H}$ if, and only if, $c = \pm 1$.

b. Since $\alpha I_2 \in \mathscr{H}$ if, and only if, α is 0 or 2, it follows that $\mathscr{H}$ is neither closed under addition, nor closed under scalar multiplication. Thus, $\mathscr{H}$ is not a subspace of $M_{2\times 2}$.

Solution to Question 10. — $\mathscr{V} = \ker\big(p \rightsquigarrow p(2) - p(1)\big)\colon \mathbb{P}_3[t] \to \mathbb{R}$, whose matrix relative to the standard bases $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3[t]$ and 1 of $\mathbb{R}$ is $\begin{pmatrix} 0 & 1 & 3 & 7 \end{pmatrix}$; so $\{1, 3t - t^2, 7t - t^3\}$ is a basis of $\mathscr{V}$.

Solution to Question 11. — a. A 4×5 matrix has at least one non-pivot column, so its columns are linearly dependent.
b. Any product of elementary matrices is invertible, and thus has linearly independent columns.
c. Every column of any matrix of an injective linear transformation is a pivot column, so the columns of such a matrix are linearly independent.

Solution to Question 12. — a. $\det(A^5A^T) = (\det A)^6 = 64$.
b. $\det(B^{-1}) = (\det A)/(\det AB) = \frac{1}{3}$.
c. $32 = \det(2A) = 2^n \det(A) = 2^{n+1}$, so $n = 4$.
d. $\det(-A) = (-1)^4 \det(A) = 2$.
e. Since A and B are invertible, $\dim \operatorname{Nul}(BC) = \dim \operatorname{Nul}(AC) = 1$, so $\det(BC) = 0$.
f. Since A and B are invertible 4×4 matrices, $\operatorname{rank}(AB) = \operatorname{rank}(A) = \operatorname{rank}(B) = 4$.

Solution to Question 13. — An LU factorization of the given matrix is

$$\begin{pmatrix} 3 & 4 & 1 \\ 9 & 11 & 5 \\ -6 & 13 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -21 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 52 \end{pmatrix},$$

via the rough work

$$\begin{matrix} -1 & 2 \\ 21 & 10 \end{matrix} \quad \rightsquigarrow \quad 52.$$

Solution to Question 14. — a. Reducing

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{p}_2 - \mathbf{p}_1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 4 \\ 2 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 3 \\ 0 & 5 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

gives

$$\mathbf{p}_1 + 3\mathbf{v}_1 = \mathbf{p}_2 - \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}.$$

b. The cosine of the (acute) angle ϑ between ℓ_1 and ℓ_2 is

$$\cos(\vartheta) = \frac{|\mathbf{v}_1^T \mathbf{v}_2|}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|} = \frac{1}{6}.$$

c. Since $0 < \cos(\vartheta) < \frac{1}{2} = \cos(\frac{1}{3}\pi)$, it follows that $\frac{1}{3}\pi < \vartheta < \frac{1}{2}\pi$. So ϑ is not between 0 and $\frac{1}{3}\pi$.

Solution to Question 15. — Let

$$\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} -9 \\ 15 \\ -1 \end{pmatrix} - \mathbf{p} = \begin{pmatrix} -10 \\ 14 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

The point on the given line which is nearest to $(-9, 15, -1)$ is

$$\mathbf{p} + \operatorname{proj}_{\mathbf{v}} \mathbf{u} = \mathbf{p} + \frac{\mathbf{v}^T\mathbf{u}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{56}{14}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -11 \\ 9 \\ 5 \end{pmatrix}.$$

Solution to Question 16. — The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{1}{2}\left\|\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}\right\| = \frac{1}{2}\left\|\begin{pmatrix} -9 \\ -3 \\ -3 \end{pmatrix}\right\| = \frac{3}{2}\left\|\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}\right\| = \frac{3}{2}\sqrt{11}.$$

Solution to Question 17. — The line is $\operatorname{Span}\{\mathbf{u}\}$, where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.$$

Solution to Question 18. — a. Since $R\colon \mathbf{e}_1 \rightsquigarrow -\mathbf{e}_1$ and $R\colon \mathbf{e}_2 \rightsquigarrow \mathbf{e}_2$, the standard matrix of R is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b. Since $S\colon \mathbf{e}_2 \rightsquigarrow \mathbf{e}_2$ and $S\colon \mathbf{e}_1 + 2\mathbf{e}_2 \rightsquigarrow \mathbf{e}_1 - 5\mathbf{e}_2$, it follows that $S\colon \mathbf{e}_1 \rightsquigarrow \mathbf{e}_1 - 7\mathbf{e}_2$, and that

$$B = \begin{pmatrix} 1 & 0 \\ -7 & 1 \end{pmatrix}$$

is the standard matrix of S
c. The standard matrix of $S^{-1} \circ R$ is

$$B^{-1}A = \begin{pmatrix} 1 & 0 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -7 & 1 \end{pmatrix}.$$

Solution to Question 19. — The volume of $T(Q)$ is $|\det(A)| = 2|k|$ times the volume of Q, or $14|k|$. Thus, the volume of $T(Q)$ is 35 if, and only if $k = \pm\frac{5}{2}$.

Solution to Question 20. — a. Let A be a square matrix. If $A\mathbf{x} = A\mathbf{y}$ for distinct $\mathbf{x}, \mathbf{y}$, then A **cannot** be invertible.
b. Let T be a linear transformation with standard matrix A. The kernel of T **must** equal the null space of A and the range of T **must** equal the column space of A.
c. If $\mathbf{w}$ is in $\operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$ then $\mathbf{w}$ **might** be in $\{\mathbf{u}, \mathbf{v}\}$.
d. Let A be a non-zero, non-invertible 3×3 matrix. If the column space of A does not form a line, then the null space of A **must** form a line.

Solution to Question 21. — First, notice that $0 = (\mathbf{u} + 2\mathbf{v})\cdot(\mathbf{u} + 3\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 5\mathbf{u}\cdot\mathbf{v} + 6\mathbf{v}\cdot\mathbf{v} = 5\mathbf{u}\cdot\mathbf{v} + 15$; thus $\mathbf{u}\cdot\mathbf{v} = -3$, so the Cauchy inequality implies that $\mathbf{u} = -3\mathbf{v}$. Thus $\|\mathbf{u} + \mathbf{v}\| = \|-2\mathbf{v}\| = 2\|\mathbf{v}\| = 2$.

Solution to Question 22. — If $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0}$ then $\gamma = 0$, for otherwise $\mathbf{w} \notin \operatorname{Span}\{\mathbf{u}, \mathbf{v}\}$. So $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$, and thus $\alpha = \beta = 0$, since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. This shows that the equation $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0}$ implies that $\alpha = \beta = \gamma = 0$; *i.e.*, that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.