Question 1. — You are given that $A = \begin{pmatrix} 4 & 12 & -2 & 8 & -16 \\ -2 & -6 & 1 & -4 & 8 \\ 5 & 15 & 1 & -18 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & -2 & -1 \\ 0 & 0 & 1 & -8 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

a. Find two distinct bases for $\mathrm{Col}(A)$.
b. Find a basis of $\mathrm{Row}(A)$.
c. Find a basis of $\mathrm{Nul}(A)$.
d. Find the dimension of $\mathrm{Nul}(A^T)$.

Question 2. — Use linear algebra to balance the chemical equation $CO + O_2 \longrightarrow CO_2$.

Question 3. — Find h, k so that the linear system

$$\begin{aligned} x \quad\quad\; + 2z &= 1 \\ -x + ky + 6z &= 3h \\ 2y + 4kz &= 2 \end{aligned}$$

has: a. infinitely many solutions; b. no solution; c. a unique solution.

Question 4. — Consider the matrix $A = \begin{pmatrix} 0 & 5 & 10 \\ -1 & 3 & 11 \end{pmatrix}$.

a. Find the reduced echelon form R of A.
b. Express A as a product of elementary matrices and R.

Question 5. — Give an LU-factorization of the matrix $A = \begin{pmatrix} -3 & 1 \\ 6 & 3 \\ -21 & 27 \end{pmatrix}$.

Question 6. — Prove that if $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{u}, \mathbf{v}$ are solutions of the equation $A\mathbf{x} = \mathbf{b}$, then $\mathbf{u} + \mathbf{v}$ is not a solution of the equation $A\mathbf{x} = \mathbf{b}$.

Question 7. — Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ have determinant 5, let B be a 3×3 matrix of determinant -6, let L be a 3×3 unit lower triangular matrix and let $C = \begin{pmatrix} g & h & i \\ 3a + 2d & 3b + 2e & 3c + 2f \\ a & b & c \end{pmatrix}$.

a. Compute $\det(C)$.
b. Compute $\det(A + C)$.
c. Compute $\det\begin{pmatrix} I & 0 \\ B^T & I + L \end{pmatrix}$.
d. Compute $\det(\mathrm{adj}(B))$.

Question 8. — a. Set up a system needed to express the $\begin{pmatrix} 5 \\ 8 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \end{pmatrix}$.

b. Use Cramer's Rule to solve the system from part a, and give the linear combination as your final answer.

Question 9. — Solve the equation $2X^T + A = (XB - C)^T$, in which A, B, C and X are $n \times n$ matrices, for the matrix X. (Assume invertibility where required.)

Question 10. — Let $\mathscr{P}$ be the plane defined by $x + ky - 10z = 5$, and let $\ell = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix}\right\}$.

a. For which values of k, if any, does the point $(-4, 6, -17)$ lie on ℓ?
b. For which values of k, if any, is ℓ parallel to $\mathscr{P}$?
c. For which values of k, if any, is ℓ orthogonal to $\mathscr{P}$?
d. For which values of k, if any, does $\mathscr{P}$ intersect the plane $4x - 12y - 40z = 12$ in a line?

Question 11. — Consider the points $A(3, 5, 0)$, $B(5, 5, -2)$ and $C(3, 8, 2)$.

a. Find the area of a parallelogram with three of its vertices at A, B and C.
b. Find an normal equation of the plane which contains the points A, B and C.
c. Find the distance between C and the line AB.

Question 12. — Consider the matrix $A = \begin{pmatrix} 6 & 2 & k & 0 \\ 3 & 1 & -2 & 0 \\ k^2 & -2 & 3 & k \\ -12 & k & 8 & 0 \end{pmatrix}$.

a. Find an expression in terms of k for $\det(A)$.
b. For which values of k, if any, is the rank of A equal to 4?

Question 13. — Find a basis for, and the dimension of, $\mathscr{H} = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in M_{2\times 2} : \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0 \right\}$.

Question 14. — Let $\mathscr{R} = \{p(x) \in \mathbb{P}_2[x] : p(1)p(0) = 0\}$.

a. Provide two polynomials in $\mathscr{R}$, neither of which is a scalar multiple of the other.
b. Is $\mathscr{R}$ closed under addition? Justify your answer.
c. Is $\mathscr{R}$ closed under scalar multiplication? Justify your answer.

Question 15. — Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + z\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$.

a. Find the standard matrix of T.
b. Give a non-zero vector in the kernel of T.
c. Is T surjective? Justify your answer.
d. Find a matrix B of rank 1 such that if $S(\mathbf{x}) = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$ then the standard matrix of $S \circ T$ is a zero matrix.

Question 16. — Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & x \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y \\ 1 \end{pmatrix}$. Is T linear? Justify your answer.

Question 17. — Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{v} \times \mathbf{w}$ has length 4 and forms an angle of $\frac{1}{3}\pi$ with the unit vector $\mathbf{u}$. Solve the equation $\mathbf{u} \cdot (\mathbf{v} \times k\mathbf{w}) + 3\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = -16$.

Question 18. — Let A be the standard matrix of a rotation about the origin in $\mathbb{R}^2$. Use determinants to show that A must be invertible, regardless of the angle of rotation.

Question 19. — Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-zero vectors in $\mathbb{R}^3$ for which $5\mathbf{u} + 8\mathbf{v} + 3\mathbf{w} = \mathbf{0}$.

a. Is it possible that $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$? Explain your answer.
b. If $\mathbf{u}$ is parallel to $\mathbf{v}$, what is the dimension of $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?
c. If A is the matrix $\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$, give a non-trivial solution of $A\mathbf{x} = \mathbf{0}$.

Question 20. — Complete each sentence with **must**, **might** or **cannot**, as appropriate.

a. If the first column of a matrix B belongs to $\mathrm{Nul}(A)$ then the columns of AB _________ be linearly dependent.

b. The partitioned matrix $\begin{pmatrix} I & A \\ A & I \end{pmatrix}$ _________ be symmetric.

c. If A is a 4×3 matrix and $A\mathbf{x} = \mathbf{b}$ has no solution, then the dimension of the null space of A _________ be zero.

d. If A, B and C are distinct points in $\mathbb{R}^3$ such that $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{0}$, then A, B and C _________ be collinear.

e. If $\{\mathbf{u}, \mathbf{v}\}$ is a basis of a linear space S, then $\{6\mathbf{u} + 3\mathbf{v}, 10\mathbf{u} + 5\mathbf{v}\}$ _________ be a basis of S.

Solution to Question 1. — a. Writing $\mathbf{a}_j$ for column j of A, two bases of the column space of A are $\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_5, \mathbf{a}_6\}$. (Any two linearly independent vectors in the columns space of A form a basis of the column space of A.)

b. The transposes of the non-zero rows of any echelon form of A form a basis of the row space of A; in this case,

$$\left\{\begin{pmatrix}1\\3\\0\\-2\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-8\\6\end{pmatrix}\right\}$$

is a basis of the row space of A.

c. From the given reduction it is plain that $\mathbf{a}_2 = 3\mathbf{a}_1$, $\mathbf{a}_4 = -2\mathbf{a}_1 - 8\mathbf{a}_3$ and $\mathbf{a}_5 = -\mathbf{a}_1 + 6\mathbf{a}_3$, so

$$\left\{\begin{pmatrix}-3\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\8\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-6\\0\\1\end{pmatrix}\right\}.$$

d. The rank formula implies that $\dim \operatorname{Nul}(A^T) = 3 - \operatorname{rank}(A) = 3 - 2 = 1$.

Solution to Question 2. — Counting Carbon, the first and last coefficients must be equal, each of which must be twice the second coefficient as is seen by counting Oxygen. Thus, by the algebra of linear combinations, the equation balances as $2CO + O_2 \longrightarrow 2CO_2$.

Solution to Question 3. — The augmented matrix of the given system is

$$\begin{pmatrix}1 & 0 & 2 & 1\\ -1 & k & 6 & 3h\\ 0 & 2 & 4k & 2\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 2 & 1\\ 0 & 1 & 2k & 1\\ 0 & 0 & 2(4-k^2) & 3h-k+1\end{pmatrix},$$

so: the system has infinitely many solutions if $k = 2$ and $h = \frac{1}{3}$ or else $k = -2$ and $h = -1$; the system has no solution if $k = 2$ and $h \neq \frac{1}{3}$ or else $k = -2$ and $h \neq -1$; the system has a unique solution if $k \neq \pm 2$, in which case h can assume any real value.

Solution to Question 4. — Reducing A gives

$$\begin{pmatrix}0 & 5 & 10\\ -1 & 3 & 11\end{pmatrix} \sim \begin{pmatrix}-1 & 3 & 11\\ 0 & 1 & 2\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -5\\ 0 & 1 & 2\end{pmatrix},$$

via the sequence of elementary row operations $R_1 \longleftrightarrow R_2$, $R_2 \leftarrow \frac{1}{5}R_2$, $R_1 \leftarrow R_1 - 3R_2$, $R_1 \leftarrow -R_1$. Thus,

$$\begin{pmatrix}0 & 5 & 10\\ -1 & 3 & 11\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -5\\ 0 & 1 & 2\end{pmatrix}.$$

Solution to Question 5. — One has

$$\begin{pmatrix}-3 & 1\\ 6 & 3\\ -21 & 27\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 7 & 4 & 1\end{pmatrix}\begin{pmatrix}-3 & 1\\ 0 & 5\\ 0 & 0\end{pmatrix}, \qquad \text{via the rough work} \qquad \begin{matrix}5\\20\end{matrix} \rightsquigarrow 0.$$

Solution to Question 6. — If $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$, then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{b} + \mathbf{b} \neq \mathbf{b}$ unless $\mathbf{b} = \mathbf{0}$.

Solution to Question 7. — a. Since

$$C = \begin{pmatrix}0 & 0 & 1\\ 3 & 2 & 0\\ 1 & 0 & 0\end{pmatrix}A, \qquad \text{and} \qquad \begin{vmatrix}0 & 0 & 1\\ 3 & 2 & 0\\ 1 & 0 & 0\end{vmatrix} = -2,$$

it follows that $\det(C) = -2\det(A) = -10$.

b. The first and third rows of $A + C$ are equal, so $\det(A + C) = 0$.

c. $\det\begin{pmatrix}I & 0\\ B^T & I+L\end{pmatrix} = 2^3 = 8$, since the matrix is triangular with three 1s and three 2s on its diagonal.

d. Since B is a 3×3 matrix, $\det(\operatorname{adj}(B)) = (\det B)^2 = 36$.

Solution to Question 8. — a. The coefficients in the linear combination are the solutions of

$$\begin{aligned} 3x + 7y &= 5\\ -4x - 3y &= 8. \end{aligned}$$

b. Cramer's Rule gives

$$x = \frac{\begin{vmatrix}5 & 7\\ 8 & -3\end{vmatrix}}{\begin{vmatrix}3 & 7\\ -4 & -3\end{vmatrix}} = -\frac{71}{19} \qquad \text{and} \qquad y = \frac{\begin{vmatrix}3 & 5\\ -4 & 8\end{vmatrix}}{\begin{vmatrix}3 & 7\\ -4 & -3\end{vmatrix}} = \frac{44}{19},$$

so that

$$-\frac{71}{19}\begin{pmatrix}3\\-4\end{pmatrix} + \frac{44}{19}\begin{pmatrix}7\\-3\end{pmatrix} = \begin{pmatrix}5\\8\end{pmatrix}.$$

Solution to Question 9. — The equation is equivalent to $2X + A^T = XB - C$, *i.e.*, $X(B - 2I) = A^T + C$, or $X = (A^T + C)(B - 2I)^{-1}$.

Solution to Question 10. — a. The point $(-4, 6, -17)$ lies on ℓ if, and only if, for some $t \in \mathbb{R}$,

$$\begin{pmatrix}2\\k\\-1\end{pmatrix} + t\begin{pmatrix}3\\1\\k\end{pmatrix} = \begin{pmatrix}-4\\6\\-17\end{pmatrix}.$$

The first entry gives $2 + 3t = -4$ or $t = -2$, and thus the second entry gives $k - 2 = 6$, or $k = 8$, which is confirmed by checking the third entry: $-1 + (-2) \cdot 8 = -17$.

b. The line ℓ is parallel to the plane $\mathscr{P}$ if, and only if, the normal to $\mathscr{P}$ is orthogonal to the direction of ℓ; *i.e.*,

$$\begin{pmatrix}1 & k & -10\end{pmatrix}\begin{pmatrix}3\\1\\k\end{pmatrix} = 0, \qquad \text{equivalently} \qquad 3 - 9k = 0, \qquad \text{or} \qquad k = \tfrac{1}{3}.$$

c. The line ℓ is orthogonal to the plane $\mathscr{P}$ if, and only if, the normal to $\mathscr{P}$ is parallel to the direction of ℓ, *i.e.*,

$$\alpha\begin{pmatrix}1\\k\\-10\end{pmatrix} = \begin{pmatrix}3\\1\\k\end{pmatrix}$$

for some $\alpha \in \mathbb{R}$. The first entry gives $\alpha = 3$, so the second entry gives $k = \frac{1}{3}$, which is contradicted by (part b or) the third entry: $3 \cdot (-10) \neq \frac{1}{3}$. Therefore, there is no value of k for which ℓ is orthogonal to $\mathscr{P}$.

d. The plane $\mathscr{P}$ intersects the plane defined by $4x - 12y - 40z = 12$ in a line if, and only if, the planes are not parallel; *i.e.*, there is no scalar α such that

$$\alpha\begin{pmatrix}1\\k\\-10\end{pmatrix} = \begin{pmatrix}4\\-12\\-40\end{pmatrix} = 4\begin{pmatrix}1\\-3\\-10\end{pmatrix}.$$

Thus, the planes intersect in a line for every value of k except -3.

Solution to Question 11. — Let

$$\mathbf{u} = \overrightarrow{AB} = 2\begin{pmatrix}1\\0\\-1\end{pmatrix}, \qquad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix}0\\3\\2\end{pmatrix} \qquad \text{and} \qquad \mathbf{n} = \tfrac{1}{2}\mathbf{u}\times\mathbf{v} = \begin{pmatrix}3\\-2\\3\end{pmatrix}.$$

a. The area of any parallelogram with three of its vertices at A, B and C, is equal to

$$\|\mathbf{u}\times\mathbf{v}\| = 2\sqrt{3^2+(-2)^2+3^2} = 2\sqrt{22}.$$

b. The plane containing A, B and C is defined by $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\,\overrightarrow{OA}$, *i.e.*, $3x_1 - 2x_2 + 3x_3 = -1$.
c. The distance between C and the line AB is equal to

$$\frac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|} = \frac{2\sqrt{22}}{2\sqrt{2}} = \sqrt{11}.$$

Solution to Question 12. — By a direct computation,

$$\det(A) = \begin{vmatrix}6&2&k&0\\3&1&-2&0\\k^2&-2&3&k\\-12&k&8&0\end{vmatrix} = -k\begin{vmatrix}6&2&k\\3&1&-2\\-12&k&8\end{vmatrix} = -k\begin{vmatrix}0&0&k+4\\3&1&-2\\0&k+4&0\end{vmatrix} = -3k(k+4)^2.$$

The rank of A is 4 if, and only if, A is non-singular; *i.e.*, if, and only if, $k \neq 0, -4$.

Solution to Question 13. — The defining condition is $w + 3y + 3x + 9z = 0$, so $\mathscr{H}$ is spanned by

$$\begin{pmatrix}-9&0\\0&1\end{pmatrix}, \begin{pmatrix}-3&0\\1&0\end{pmatrix}, \begin{pmatrix}-3&1\\0&0\end{pmatrix},$$

which are linearly independent and thus form a basis of $\mathscr{H}$. The dimension of $\mathscr{H}$ is 3.

Solution to Question 14. — a. Plainly $x, x-1 \in \mathscr{R}$, neither of which is a scalar multiple of the other.

b. Since $x, x-1 \in \mathscr{R}$ and $p(x) = x + x - 1 = -1 \notin \mathscr{R}$ (for $p(1)p(0) = -1 \neq 0$), it follows that $\mathscr{R}$ is not closed under addition.

c. If $q(x) \in \mathscr{R}$ and $\alpha \in \mathbb{R}$, then $(\alpha q)(1)(\alpha q)(0) = \alpha^2 q(1)q(0) = \alpha^2 0 = 0$, so $(\alpha q)(x) \in \mathscr{R}$. Therefore, $\mathscr{R}$ is closed under scalar multiplication.

Solution to Question 15. — a. Since

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = x\begin{pmatrix}-3\\1\\2\end{pmatrix} + y\begin{pmatrix}-1\\0\\1\end{pmatrix} + z\begin{pmatrix}-2\\2\\0\end{pmatrix} = \begin{pmatrix}-3&-1&-2\\1&0&2\\2&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}, \qquad A = \begin{pmatrix}-3&-1&-2\\1&0&2\\2&1&0\end{pmatrix}$$

is the standard matrix of T.

b. Since

$$A \sim \begin{pmatrix}-3&-1&-2\\0&-\frac{1}{3}&\frac{4}{3}\\0&\frac{1}{3}&-\frac{4}{3}\end{pmatrix} \sim \begin{pmatrix}-3&0&-6\\0&1&-4\\0&0&0\end{pmatrix} \sim \begin{pmatrix}1&0&2\\0&1&-4\\0&0&0\end{pmatrix}, \qquad \text{it follows that} \qquad \begin{pmatrix}-2\\4\\1\end{pmatrix}$$

is a non-zero vector in the kernel of T (*i.e.*, the null space of A).

c. The rank of A is 2, not 3, so T is not surjective.

d. Since

$$\begin{pmatrix}1&1&1\end{pmatrix}\begin{pmatrix}-3&-1&-2\\1&0&2\\2&1&0\end{pmatrix} = \begin{pmatrix}0&0&0\end{pmatrix},$$

$B = \begin{pmatrix}1&1&1\end{pmatrix}$ is a matrix as required.

Solution to Question 16. — Since $T\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1&0\\2&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$, it follows that T is not linear.

Solution to Question 17. — Since

$$\mathbf{u}\cdot(\mathbf{v}\times k\mathbf{w}) + 3\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = (k-3)\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (k-3)\|\mathbf{u}\|\|\mathbf{v}\times\mathbf{w}\|\cos\left(\tfrac{1}{3}\pi\right) = (k-3)\cdot 4\cdot\tfrac{1}{2} = 2(k-3),$$

this expression is equal to -16 if, and only if, $k-3=-8$, or $k=-5$.

Solution to Question 18. — Since the mapping $\mathbf{x} \rightsquigarrow A\mathbf{x}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation, it preserves area and orientation, so its determinant is $1 \neq 0$. Therefore, A is invertible.

Solution to Question 19. — a. Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent, they cannot span $\mathbb{R}^3$.

b. If $\mathbf{u}$ is parallel to $\mathbf{v}$ then so is $\mathbf{w} = -\frac{5}{3}\mathbf{u} - \frac{8}{3}\mathbf{v}$, so the dimension of $\operatorname{Span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is equal to 1.

c. The relation $5\mathbf{u} + 8\mathbf{v} + 3\mathbf{w} = \mathbf{0}$ implies that $\begin{pmatrix}5\\8\\3\end{pmatrix}$ is a solution of $\begin{pmatrix}\mathbf{u}&\mathbf{v}&\mathbf{w}\end{pmatrix}\mathbf{x} = \mathbf{0}$.

Solution to Question 20. — a. If the first column of a matrix B belongs to $\operatorname{Nul}(A)$ then the columns of AB **must** be linearly dependent.

b. The partitioned matrix $\begin{pmatrix}I&A\\A&I\end{pmatrix}$ **might** be symmetric.

c. If A is a 4×3 matrix and $A\mathbf{x} = \mathbf{b}$ has no solution, then the dimension of the null space of A **might** be zero.

d. If A, B and C are distinct points in $\mathbb{R}^3$ such that $\overrightarrow{AB}\times\overrightarrow{AC} = \mathbf{0}$, then A, B and C **must** be collinear.

e. If $\{\mathbf{u},\mathbf{v}\}$ is a basis of a linear space S, then $\{6\mathbf{u}+3\mathbf{v}, 10\mathbf{u}+5\mathbf{v}\}$ **cannot** be a basis of S.