Question 1. — Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{pmatrix}$ be a 4×5 matrix and $\mathbf{b} \in \mathbb{R}^4$ be such that

$$\begin{pmatrix} A & \mathbf{b} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & -4 & 10 \\ 0 & 1 & 4 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 & -8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a. Write the general solution of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
b. Circle the lists below which are bases of $\mathrm{Col}(A)$.

$\{\mathbf{a}_3, \mathbf{a}_5\}$ $\quad$ $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ $\quad$ $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

$\{\mathbf{a}_1, 2\mathbf{a}_2, 5\mathbf{a}_5\}$ $\quad$ $\{\mathbf{a}_3, 2\mathbf{a}_3 + \mathbf{a}_5\}$ $\quad$ $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

c. What is $\dim(\mathrm{Nul}(A^T))$?
d. What is $\mathrm{rank}(A^T)$?
e. Given that $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 16 \\ 13 \\ 1 \\ 31 \end{pmatrix}$, compute $\mathbf{a}_3$ and $\mathbf{a}_4$.

Question 2. — Let $A = \begin{pmatrix} 5 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

a. Find A^{-1}.
b. Write A^{-1} as a product of elementary matrices.

Question 3. — Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ k+2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 3 \\ k+3 \\ -6 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix}$.

a. For which values of k is $\mathbf{v}_4 \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
b. For which values of k is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly indepdnent?

Question 4. — Let $A = \begin{pmatrix} 0 & 2 & 1 & 1 \\ -2 & 1 & 1 & 3 \\ 4 & 4 & 0 & 3 \\ 2 & 1 & -2 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.
Use Cramer's rule to solve $A\mathbf{x} = \mathbf{b}$ for x_2 only.

Question 5. — Given 3×3 matrices A and B with $\det(A) = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 5$ and $\det(B) = -6$, evaluate each of the following.

a. $\begin{vmatrix} 3x & 3y & 3z \\ -a & -b & -c \\ a+2p & b+2q & c+2r \end{vmatrix}$ b. $\det\big((2B)^{-1}\big)$ c. $\det(3B^T A)$ d. $\mathrm{rank}(BA)$

Question 6. — Solve $(3A^T B^{-1})^T X (6BA)^{-1} = B^{-1}$ for X, given that A and B are invertible matrices of the same size.

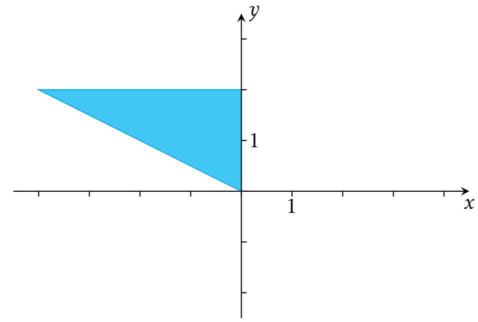
Question 7. — Given $B = \begin{pmatrix} x & y & -5 \\ y & z & 5 \\ 4 & 1 & 10 \end{pmatrix}$ and $\mathrm{adj}(B) = \begin{pmatrix} 5 & 15 & -5 \\ 40 & 50 & -5 \\ -6 & -11 & -1 \end{pmatrix}$, find x, y and z.

Question 8. — Let $\mathscr{L}$ be the line given by $\begin{pmatrix} 1 \\ h \\ k+1 \end{pmatrix} + t \begin{pmatrix} h-2 \\ 3 \\ k \end{pmatrix}$, and let $\mathscr{P}$ be the plane defined by $3x + 2y + z = 5$. Find conditions on h and k so that:

a. $\mathscr{L}$ is perpendicular to $\mathscr{P}$; b. $\mathscr{L}$ is contained in $\mathscr{P}$.

Question 9. — Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the rotation about the origin by π, and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the vertical expanstion by a factor of 2. Compute the standard matrix of $T \circ R$.

Question 10. — Give the matrix of a shear $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ so that the image of the triangle below is an isosceles triangle.



Question 11. — Consider the points $A(2,1,0)$, $B(2,4,4)$ and $C(0,-2,6)$.

a. Find a normal equation of the plane containing A, B and C.
b. Find the area of triangle ABC.
c. Find the cosine of the angle at the vertex A of triangle ABC.
d. Find the distance between the point C and the line AB.

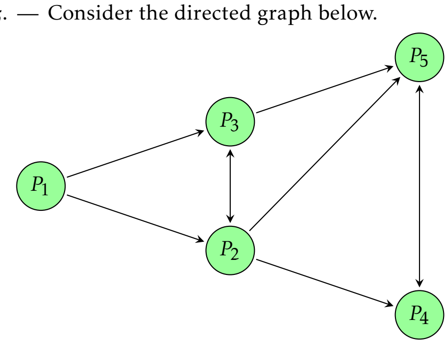
Question 12. — Let $W = \{p \in \mathbb{P}_3[x]\colon p(1) = 0 \text{ and } p'(1) = p(-1)\}$. Find a basis of W.

Question 13. — Prove that $\mathrm{Nul}(B) \subset \mathrm{Nul}(AB)$, whenever AB is defined.

Question 14. — Let $W = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_{2\times 2}\colon ad = b^2 - c^2 \right\}$.

a. Give two matrices in W, neither of which is a multiple of the other.
b. Is W closed under addition? Justify your answer.
c. Is W closed under scalar multiplication? Justify your answer.

Question 15. — Consider the directed graph below.



a. Find the adjacency matrix M of the graph.
b. Compute M^2 and M^4.
c. How many walks of length 4 end at P_5?
d. How many closed walks of length 8 are there in total?

Question 16. — Let $\mathbf{a}$ and $\mathbf{b}$ be mutually orthogonal unit vectors in $\mathbb{R}^3$, and let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $T(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})\mathbf{b} + (\mathbf{b} \cdot \mathbf{x})\mathbf{a}$.

a. Simplify $T(\mathbf{a})$ as much as possible.
b. Simplify $T(\mathbf{a} \times \mathbf{b})$ as much as possible.
c. Is T injective (one-to-one)? Explain.

a. Write the general solution of $A\mathbf{x}=\mathbf{b}$ in parametric vector form.
b. Circle the lists below which are bases of $\mathrm{Col}(A)$.

$$\{\mathbf{a}_3,\mathbf{a}_5\} \qquad \{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_4\} \qquad \{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$$
$$\{\mathbf{a}_1,2\mathbf{a}_2,5\mathbf{a}_5\} \qquad \{\mathbf{a}_3,2\mathbf{a}_3+\mathbf{a}_5\} \qquad \{\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}$$

c. What is $\dim(\mathrm{Nul}(A^T))$?
d. What is $\mathrm{rank}(A^T)$?
e. Given that $\mathbf{a}_1=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $\mathbf{a}_2=\begin{pmatrix}2\\1\\-1\\1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}16\\13\\1\\31\end{pmatrix}$, compute $\mathbf{a}_3$ and $\mathbf{a}_4$.

Solution to Question 1. — a. A particular solution of the equation is

$$\mathbf{p}=\begin{pmatrix}10\\3\\0\\6\\0\end{pmatrix}, \text{ and Nul}(A) \text{ is generated by } \mathbf{u}=\begin{pmatrix}-2\\-4\\1\\0\\0\end{pmatrix} \text{ and } \mathbf{v}=\begin{pmatrix}4\\-5\\0\\8\\1\end{pmatrix},$$

so the general solution of $A\mathbf{x}=\mathbf{b}$ is $\mathbf{p}+\mathrm{Span}\{\mathbf{u},\mathbf{v}\}$.

b. The column space of A is three dimensional, so any three linearly independent columns of A (or any three linearly independent vectors in the column space for that matter) will form a basis of $\mathrm{Col}(A)$. Thus,

$$\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_4\}, \quad \{\mathbf{a}_1,2\mathbf{a}_2,5\mathbf{a}_5\}, \quad , \quad \{\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4\}$$

are all bases of $\mathrm{Col}(A)$.

c. The column space of A is a 3-dimensional subspace of $\mathbb{R}^4$, so the dimension of the null space of A^T is $4-3=1$.

d. The rank of A^T is equal to the rank of A, which is 3.

e. From the reduced echelon form of A, it follows that

$$\mathbf{a}_3=2\mathbf{a}_1+4\mathbf{a}_2=2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+4\begin{pmatrix}2\\1\\-1\\1\end{pmatrix}=\begin{pmatrix}10\\6\\-2\\6\end{pmatrix}$$

and $10\mathbf{a}_1+3\mathbf{a}_2+6\mathbf{a}_4=\mathbf{b}$, so

$$6\mathbf{a}_4=\mathbf{b}-10\mathbf{a}_1-3\mathbf{a}_2=\begin{pmatrix}16\\13\\1\\31\end{pmatrix}-10\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-3\begin{pmatrix}2\\1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\-6\\18\end{pmatrix}, \quad \text{or} \quad \mathbf{a}_4=\begin{pmatrix}0\\0\\-1\\3\end{pmatrix}.$$

Solution to Question 2. — a. Applying the elementary row operations $R_2\leftrightarrow R_3$, $R_2\leftarrow R_2-3R_3$, $R_1\leftarrow R_1-5R_2$ and $R_1\leftarrow\frac{1}{5}R_1$ to A

$$A\sim\begin{pmatrix}5&5&0\\0&1&3\\0&0&1\end{pmatrix}\sim\begin{pmatrix}5&5&0\\0&1&0\\0&0&1\end{pmatrix}\sim\begin{pmatrix}5&0&0\\0&1&0\\0&0&1\end{pmatrix}\sim I_3, \quad \text{so}$$

$$I_3\sim\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\sim\begin{pmatrix}1&0&0\\0&-3&1\\0&1&0\end{pmatrix}\sim\begin{pmatrix}1&15&-5\\0&-3&1\\0&1&0\end{pmatrix}\sim\begin{pmatrix}\frac{1}{5}&3&-1\\0&-3&1\\0&1&0\end{pmatrix}=A^{-1}.$$

b. Writing E_1, E_2, E_3, E_4 for the elementary matrices associated to the elemntary row operations used to reduce A to I gives

$$A^{-1}=E_4E_3E_2E_1=\begin{pmatrix}\frac{1}{5}&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&-5&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-3\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Solution to Question 3. — a. Reducing

$$\begin{pmatrix}\mathbf{v}_1&\mathbf{v}_2&\mathbf{v}_3&\mathbf{v}_4\end{pmatrix}\sim\begin{pmatrix}1&2&3&5\\0&-1&k&1\\0&k&-9&3\end{pmatrix}\sim\begin{pmatrix}1&2&3&5\\0&-1&k&1\\0&0&k^2-9&k+3\end{pmatrix}$$

gives $\mathbf{v}_4\in\mathrm{Span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ if, and only if, $k\neq 3$.

b. If $k\neq\pm 3$ then $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly indpendent.

Solution to Question 4. — First of all

$$\det(A)=\begin{vmatrix}0&0&0&1\\-2&-5&-2&3\\4&-2&-3&3\\2&-5&-5&3\end{vmatrix}=-\begin{vmatrix}-2&-5&-2\\0&-12&-7\\0&-10&-7\end{vmatrix}=28\begin{vmatrix}6&1\\5&1\end{vmatrix}=28,$$

and

$$\det A_2(\mathbf{b})=\begin{vmatrix}0&0&1&1\\-2&1&1&3\\4&0&0&3\\2&0&-2&3\end{vmatrix}=\begin{vmatrix}0&0&1\\4&-3&3\\2&-5&3\end{vmatrix}=-2\begin{vmatrix}2&3\\1&5\end{vmatrix}=-14,$$

so Cramer's rule gives $x_2=\dfrac{\det A_2(\mathbf{b})}{\det(A)}=-\frac{1}{2}$.

Solution to Question 5. — a. Direct calculation gives

$$\begin{vmatrix}3x&3y&3z\\-a&-b&-c\\a+2p&b+2q&c+2r\end{vmatrix}=\begin{vmatrix}0&0&3\\-1&0&0\\1&2&0\end{vmatrix}\begin{vmatrix}a&b&c\\p&q&r\\x&y&z\end{vmatrix}=-6\cdot 5=-30.$$

b. Since B is a 3×3 matrix, $\det\big((2B)^{-1}\big)\big(\det(2B)\big)^{-1}=2^{-3}\cdot\big(-\frac{1}{6}\big)=-\frac{1}{48}$.

c. Since A and B are 3×3 matrices, $\det(3B^TA)=3^3\cdot(-6)\cdot 5=-810$.

d. Since A and B are non-singular 3×3 matrices, the rank of BA is 3.

Solution to Question 6. — Direct multiplication gives

$$X=\Big(\big(3A^TB^{-1}\big)^T\Big)^{-1}B^{-1}(6BA)=\tfrac{1}{3}A^{-1}B^TB^{-1}6BA=2A^{-1}B^TA.$$

Solution to Question 7. — Write b'_{ij} for the entries of $\mathrm{adj}(B)$ and B_{ij} for the matrix obtained by deleting row i and column j of B. Then

$$50=b'_{22}=\det(B_{22})=\begin{vmatrix}x&-5\\4&10\end{vmatrix}=10(x+2) \qquad \text{so} \qquad x=3,$$

$$15=b'_{12}=-\det(B_{21})=-\begin{vmatrix}y&-5\\1&10\end{vmatrix}=-5(2y+1) \qquad \text{so} \qquad y=-2$$

and

$$5=b'_{11}=\det(B_11)=\begin{vmatrix}z&5\\1&10\end{vmatrix}=10z-5 \qquad \text{so} \qquad z=1.$$

Solution to Question 8. — a. $\mathscr{L}$ is perpendicular to $\mathscr{P}$ precisely when there is a scalar α such that

$$\begin{pmatrix}h-2\\3\\k\end{pmatrix}=\alpha\begin{pmatrix}3\\2\\1\end{pmatrix}, \quad \textit{i.e.}, \quad \alpha=\tfrac{2}{3}, \quad \text{so} \quad \tfrac{2}{3}(h-2)=3 \quad \text{and} \quad \tfrac{2}{3}k=1;$$

thus, $h=\frac{13}{2}$ and $k=\frac{3}{2}$.

b. $\mathscr{L}$ is contained in $\mathscr{P}$ precisely when

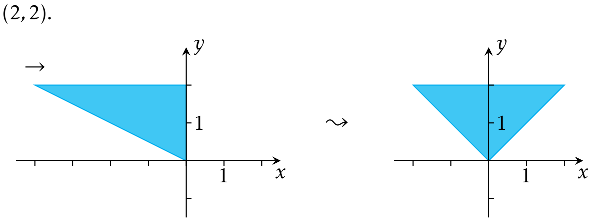
$$0=\begin{pmatrix}3&2&1\end{pmatrix}\begin{pmatrix}h-2\\3\\k\end{pmatrix}=3h+k \quad \text{and} \quad 5=\begin{pmatrix}3&2&1\end{pmatrix}\begin{pmatrix}1\\h\\k+1\end{pmatrix}=2h+k+4.$$

The first equation gives $k=-3h$, and the second equation then becomes $-h+4=5$, so $h=-1$ and $k=3$.

Solution to Question 9. — The standard matrix of a rotation about the origin by π is $[R]=-I$ and the standard matrix of a vertical expansion by a factor of 2 is $[T]=\begin{pmatrix}\mathbf{e}_1&2\mathbf{e}_2\end{pmatrix}$, so the standard matrix of the composite $T\circ R$ is

$$[T][R]=\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=\begin{pmatrix}-1&0\\0&-2\end{pmatrix}.$$

Solution to Question 10. — One possibility is the horizontal shear which maps the segment joining $(-4,2)$ and $(0,2)$ to the segment joining $(-2,2)$ and $(2,2)$.

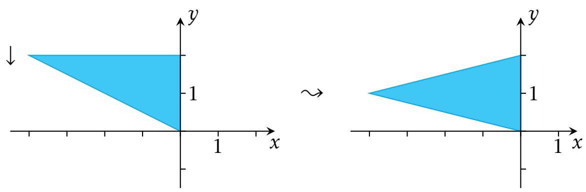


In this case the side of the image joining $(-2,2)$ to $(0,0)$ has the same length as the side joining $(2,2)$ to $(0,0)$; namely, $2\sqrt{2}$. The standard matrix of this shear satisfies

$$\begin{pmatrix}-2\\2\end{pmatrix}=\begin{pmatrix}1&a\\0&1\end{pmatrix}\begin{pmatrix}-4\\2\end{pmatrix}=\begin{pmatrix}-4+2a\\2\end{pmatrix},\quad \text{so } a=1. \qquad \text{Thus,} \quad \begin{pmatrix}1&1\\0&1\end{pmatrix}$$

is the standard matrix of the shear.

Another possibility is the vertical shear which maps $(-4,2)$ to $(-4,1)$.



In this case the side of the image joining $(-4,1)$ to $(0,2)$ has the same length as the side joining $(-4,1)$ to $(0,0)$; namely, $\sqrt{17}$. The standard matrix of this shear satisfies

$$\begin{pmatrix}-4\\1\end{pmatrix}=\begin{pmatrix}1&0\\a&1\end{pmatrix}\begin{pmatrix}-4\\2\end{pmatrix}=\begin{pmatrix}-4\\-4a+2\end{pmatrix},\quad \text{so } a=\tfrac{1}{4}. \qquad \text{Thus,} \quad \begin{pmatrix}1&0\\\frac{1}{4}&1\end{pmatrix}$$

is the standard matrix of the shear.

Solution to Question 11. — Let

$$\mathbf{u}=\overrightarrow{AB}=\begin{pmatrix}0\\3\\4\end{pmatrix},\ \mathbf{v}=\overrightarrow{AC}=\begin{pmatrix}-2\\-3\\6\end{pmatrix} \text{ and } \mathbf{n}=\tfrac{1}{2}\mathbf{u}\times\mathbf{v}=\tfrac{1}{2}\begin{pmatrix}0\\3\\4\end{pmatrix}\times\begin{pmatrix}-2\\-3\\6\end{pmatrix}=\begin{pmatrix}15\\-4\\3\end{pmatrix}.$$

a. The plane containing A, B and C is defined by the normal equation $\mathbf{n}^T\mathbf{x}=\mathbf{n}^T\mathbf{a}$, or $15x_1-4x_2+3x_3=26$.

b. The area of triangle ABC is

$$\tfrac{1}{2}|\mathbf{u}\times\mathbf{v}|=|\mathbf{n}|=\sqrt{225+16+9}=\sqrt{250}=5\sqrt{10}.$$

c. The cosine of the angle at the vertex A of triangle ABC is

$$\frac{\mathbf{u}^T\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}=\frac{15}{5\cdot 7}=\frac{3}{7}.$$

d. The distance between C and the line AB is equal to the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ divided by the length of $\mathbf{u}$, which is

$$\frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}|}=\frac{2\cdot 5\sqrt{10}}{5}=2\sqrt{10}.$$

Solution to Question 12. — The subspace W of $\mathbb{P}_3[x]$ is the kernel of the linear map $p\rightsquigarrow\begin{pmatrix}p(1)\\p(-1)-p'(1)\end{pmatrix}$. The standard matrix (that is, relative to the bases $1,x,x^2,x^3$ of $\mathbb{P}_3[x]$ and $\mathbf{e}_1,\mathbf{e}_2$ of $\mathbb{R}^2$) of this linear map is

$$A=\begin{pmatrix}1&1&1&1\\1&-2&-1&-4\end{pmatrix}\sim\begin{pmatrix}1&0&\frac{1}{3}&-\frac{2}{3}\\0&1&\frac{2}{3}&\frac{5}{3}\end{pmatrix},\quad \text{so} \quad \begin{pmatrix}-1\\-2\\3\\0\end{pmatrix},\begin{pmatrix}2\\-5\\0\\3\end{pmatrix}$$

generate the null space A, and $-1-2x+3x^2, 2-5x+3x^3$ is a basis of W.

Solution to Question 13. — If $\mathbf{x}\in\mathrm{Nul}(B)$ then $B\mathbf{x}=\mathbf{0}$, so $AB\mathbf{x}=A\mathbf{0}=\mathbf{0}$, so $\mathbf{x}\in\mathrm{Nul}(AB)$. Therefore, $\mathrm{Nul}(B)\subset\mathrm{Nul}(AB)$.

Solution to Question 14. — a. Let

$$A=\begin{pmatrix}1&1\\1&0\end{pmatrix} \qquad \text{and} \qquad B=\begin{pmatrix}0&1\\1&1\end{pmatrix};$$

then in each case $ad=0$ ($1\cdot 0$ in A and $0\cdot 1$ in B) and $b^2-c^2=1^2-1^2=0$, so $A,B\in W$, neither of which is a scalar multiple of the other.

b. The matrices A and B from part a belong to W, but

$$A+B=\begin{pmatrix}1&2\\2&1\end{pmatrix}\notin W, \qquad \text{since} \qquad 1\cdot 1\neq 2^2-2^2.$$

Therefore, W is not closed under addition.

c. If $\alpha\in\mathbb{R}$ and

$$M=\begin{pmatrix}a&c\\b&d\end{pmatrix}\in W, \qquad \text{then} \qquad \alpha M=\begin{pmatrix}\alpha a&\alpha c\\\alpha b&\alpha d\end{pmatrix}\in W,$$

since $(\alpha a)(\alpha d)=\alpha^2 ad=\alpha^2(b^2-c^2)=(\alpha b)^2-(\alpha c)^2$. Therefore, W is closed under scalar multiplication.

Solution to Question 15. — a. The adjacency matrix of the graph is

$$M=\begin{pmatrix}0&1&1&0&0\\0&0&1&1&1\\0&1&0&0&1\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}$$

b. Direct calculations give

$$M^2=\begin{pmatrix}0&1&1&1&2\\0&1&0&1&2\\0&0&1&2&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \qquad \text{and} \qquad M^4=\begin{pmatrix}0&1&1&4&5\\0&1&0&2&4\\0&0&1&4&2\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

c. The number of walks of length 4 which end at P_5 is equal to the sum of the entries in column 5 of M^4, which is $5+4+2+0+1=12$.

d. Since (by direct calculation)

$$M^8=\begin{pmatrix}0&1&1&10&11\\0&1&0&4&8\\0&0&1&8&4\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

the number of closed walks of length 8 is $0+1+1+1+1=4$ ($=\mathrm{trace}(M^8)$).

Solution to Question 16. — a. Since $\mathbf{a}$ and $\mathbf{b}$ are mutually orthogonal unit vectors, $\mathbf{a}^T\mathbf{a}=\mathbf{b}^T\mathbf{b}=1$ and $\mathbf{a}^T\mathbf{b}=\mathbf{b}^T\mathbf{a}=0$, so

$$T(\mathbf{a})=(\mathbf{a}^T\mathbf{a})\mathbf{b}+(\mathbf{b}^T\mathbf{a})\mathbf{a}=1\mathbf{b}+0\mathbf{a}=\mathbf{b}.$$

b. Since $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, it follows that

$$T(\mathbf{a}\times\mathbf{b})=(\mathbf{a}^T(\mathbf{a}\times\mathbf{b}))\mathbf{b}+(\mathbf{b}^T(\mathbf{a}\times\mathbf{b}))\mathbf{a}=0\mathbf{b}+0\mathbf{a}=\mathbf{0}.$$

c. Since $\mathbf{a},\mathbf{b}$ are mutually orthogonal unit vectors, $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|=1\neq 0$, so $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$. But $T(\mathbf{a}\times\mathbf{b})=\mathbf{0}$ by part b, so T is not injective.