Question 1. — Let $A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{pmatrix}$ be the matrix

$$A = \begin{pmatrix} 2 & -6 & 3 & 1 & 2 \\ -3 & 9 & 4 & 5 & -5 \\ -1 & 3 & 7 & 6 & -3 \\ 1 & -3 & 5 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and let $\mathbf{b} = 3\mathbf{a}_2 + \mathbf{a}_3 - 2\mathbf{a}_5$.

a. Give a basis of the null space of A.
b. Find a particular vector $\mathbf{p}$ for which $A\mathbf{p} = \mathbf{b}$.
c. Write the solution of $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
d. Give a basis of the row space of A.
e. What is the dimension of the null space of A^T?

Question 2. — Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 3 \end{pmatrix}$.

a. Find A^{-1}.
b. Use A^{-1} to solve the equation $YA = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ for Y.

Question 3. — Consider the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & 2x_3 & = & k+5 \\ -x_1 & + & (h-1)x_2 & + & (h^2-6)x_3 & = & k-1 \\ 3x_1 & + & 9x_2 & + & (h^2-h)x_3 & = & k^2+3k+11 \end{array}$$

a. For which pairs h, k does the system have no solution?
b. For which pairs h, k does the system have a unique solution?
c. For which pairs h, k is the solution of the system a line?
d. For which pairs h, k is the solution of the system a plane?

Question 4. — Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be linearly independent vectors. Find k so that the vectors

$$\mathbf{u} + \mathbf{v} + \mathbf{w}, \qquad \mathbf{u} + 2\mathbf{v} + k\mathbf{w} \qquad \text{and} \qquad -\mathbf{u} + \mathbf{v} + k\mathbf{w}$$

are linearly dependent.

Question 5. — Find the quadratic polynomial whose graph contains the points $(1,1)$, $(2,-9)$ and $(-1,-3)$.

Question 6. — a. Find the standard matrix of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which first performs a vertical expansion by a factor of 4, then reflects vectors in the line $x_2 = -x_1$, and finally performs a horizontal shear which maps $\mathbf{e}_2$ to $-3\mathbf{e}_1 + \mathbf{e}_2$.

b. Find the standard matrix of a linear transformation which maps the line $\begin{pmatrix} 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ onto a vertical line.

Question 7. — a. Solve the equation $X^T A + B = (CX)^T - B$ for X.
b. Find X from part a if $A = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 5 \\ 3 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 3 \\ 1 & 3 \end{pmatrix}$.

Question 8. — Let A be an $n \times n$ matrix such that $A^2 = A$.

a. Find the possible values of $\det(A)$.
b. Show that if $\det(A) \neq 0$ then $A = I$.

Question 9. — Let A, B and C be 4×4 matrices such that $\operatorname{rank}(A) = 2$, $\det(B) = -2$ and $\det(C) = 3$. Find:

a. $\det(A)$; b. $\det(-3B^3C^{-2})$; c. $\det\left(A^TC^{-1} + (BA)^T\right)$.

Question 10. — Find a basis of $\{p(x) \in \mathbb{P}_3[x] : p(1) = p(2) \text{ and } p(3) = 0\}$.

Question 11. — If A is a 8×7 matrix and the dimension of $\operatorname{Nul}(A^T)$ is 3 then the rank of A is ________ and the dimension of $\operatorname{Nul}(A)$ is ________.

Question 12. — Prove that if A and B are invertible $n \times n$ matrices and $(AB)^2 = A^2B^2$ then $AB = BA$.

Question 13. — Let ℓ be the line given by $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$ and let $\wp$ be the plane defined by $2x_1 + 3x_2 + 3x_3 = 4$.

a. The line ℓ and the plane $\wp$ are (circle the correct answer):
 – parallel.
 – perpendicular.
 – neither parallel nor perpendicular.

b. Circle the correct statement:
 – ℓ intersects $\wp$ in exactly one point.
 – ℓ does not intersect $\wp$.
 – ℓ is contained in $\wp$.

Question 14. — Given the lines $\ell_1 : \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\ell_2 : \begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

a. Find the point on ℓ_1 which is closest to the origin.
b. Compute the distance between ℓ_1 and ℓ_2.

Question 15. — Given $A(1,-1,2)$, $B(2,1,3)$, $C(4,0,4)$ and $D(-3,3,3)$.

a. Find the area of triangle ABC.
b. Find a normal equation of the plane which contains D and is parallel to the plane containing triangle ABC.
c. Compute the cosine of the angle A in triangle ABC.

Question 16. — Let P_1 be the plane defined by $2x_1 - 3x_2 + 4x_3 = 7$, let P_2 be the plane defined by $x_1 - 2x_2 + x_3 = 3$, and let ℓ be the line of intersection of the planes P_1 and P_2.

a. Find a parametric vector equation of ℓ.
b. Find a normal equation of the plane which is orthogonal to both P_1 and P_2 and contains the point $(1,1,1)$.
c. Give a parametric vector equation of the line which is parallel to both planes P_1 and P_2 and contains the point $(1,2,3)$.
d. Compute the distance between the P_1 and the point $(1,1,1)$.

Question 17. — Use linear algebra to balance the chemical equation:

$$C_4H_{10} + O_2 \to CO_2 + H_2O$$

Question 18. — Let $S = \{A \in M_{2\times 2} : \det(A) \geqslant 0\}$.

a. Is S closed under addition? Justify.
b. Is S closed under scalar multiplication? Justify.

Question 19. — Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$ such that $\|\mathbf{u}\| = 4$, $\|\mathbf{v}\| = \sqrt{3}$ and $\mathbf{u}^T\mathbf{v} = -6$.

a. What is the angle between $\mathbf{u}$ and $\mathbf{v}$? Give a simplified answer.
b. For which values of t, if any, is the angle between $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} + t\mathbf{v}$ acute? (An acute angle is an angle between 0 and $\frac{1}{2}\pi$.)

Solution to Question 1. — a. $\begin{pmatrix}3\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\\-1\\1\end{pmatrix}$ is a basis of the null space of A.

b. $\mathbf{p} = \begin{pmatrix}0\\3\\1\\0\\-2\end{pmatrix}$ is a particular solution of $A\mathbf{x} = \mathbf{b}$.

c. The solution of $A\mathbf{x} = \mathbf{b}$ is $\begin{pmatrix}0\\3\\1\\0\\-2\end{pmatrix} + s\begin{pmatrix}3\\1\\0\\0\\0\end{pmatrix} + t\begin{pmatrix}-2\\0\\1\\-1\\1\end{pmatrix}$, where $s, t \in \mathbb{R}$.

d. $\begin{pmatrix}1\\-3\\0\\0\\2\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}$ is a basis of $\operatorname{Row}(A) = \operatorname{Col}(A^T)$.

e. $\dim\left(\operatorname{Nul}(A^T)\right) = 1$.

Solution to Question 2. — a. $A^{-1} = \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & \frac{2}{3} & \frac{1}{3}\end{pmatrix}$

b. $Y = \begin{pmatrix}1 & 2 & 3\\2 & 3 & 4\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & \frac{2}{3} & \frac{1}{3}\end{pmatrix} = \begin{pmatrix}2 & 3 & 1\\3 & \frac{14}{3} & \frac{4}{3}\end{pmatrix}$.

Solution to Question 3. — The augmented matrix of the system is

$$\left(\begin{array}{cccc}1 & 3 & 2 & k+5\\-1 & h-1 & h^2-6 & k-1\\3 & 9 & h^2-h & k^2+3k+11\end{array}\right) \sim \left(\begin{array}{cccc}1 & 3 & 2 & k+5\\0 & h+2 & (h+2)(h-2) & 2(k+2)\\0 & 0 & (h+2)(h-3) & (k+2)(k-2)\end{array}\right).$$

a. There is no solution if $h = -2$ and $k \neq -2$, or else $h = 3$ and $k \neq \pm 2$.
b. There is a unique solution if $h \neq -2, 3$ and k is any real number.
c. The solution is a line if $h = 3$ and $k = \pm 2$.
d. The solution is a plane if $h = -2$ and $k = -2$.

Solution to Question 4. — $\{\mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{u}+2\mathbf{v}+k\mathbf{w}, -\mathbf{u}+\mathbf{v}+k\mathbf{w}\} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}A$, where $A = \begin{pmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & k & k\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1\\0 & 1 & 2\\0 & 0 & 3-k\end{pmatrix}$, so $k = 3$.

Solution to Question 5. — Reducing $\begin{pmatrix}1 & 1 & 1 & 1\\1 & 2 & 4 & -9\\1 & -1 & 1 & -3\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & 3\\0 & 1 & 0 & 2\\0 & 0 & 1 & -4\end{pmatrix}$ gives $p(x) = 3 + 2x - 4x^2$.

Solution to Question 6. — a. $\underbrace{\begin{pmatrix}1 & -3\\0 & 1\end{pmatrix}}_{\text{shear}}\underbrace{\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}}_{\text{reflection}}\underbrace{\begin{pmatrix}1 & 0\\0 & 4\end{pmatrix}}_{\text{expansion}} = \begin{pmatrix}3 & -4\\-1 & 0\end{pmatrix}$

b. $\mathbf{x} \rightsquigarrow \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\mathbf{x}$ maps the given line onto the line generated by $\begin{pmatrix}0\\1\end{pmatrix}$.

Solution to Question 7. — a. Transposing gives $A^T X + B^T = CX - B^T$, or $(C - A^T)X = 2B^T$, so $X = 2(C - A^T)^{-1}B^T$.

b. $X = 2\left[\begin{pmatrix}-2 & 3\\1 & 3\end{pmatrix} - \begin{pmatrix}1 & 5\\2 & 1\end{pmatrix}^T\right]^{-1}\begin{pmatrix}-3 & 5\\3 & 1\end{pmatrix}^T = \begin{pmatrix}11 & -5\\27 & -9\end{pmatrix}$.

Solution to Question 8. — a. $(\det(A))^2 = \det(A)$, so $\det(A) = 0$ or 1.
b. $A(A - I) = 0$, so either $A = I$ or else A is singular.

Solution to Question 9. — a. $\det(A) = 0$ since the rank of A is < 4.
b. $\det(-3B^3C^{-2}) = (-3)^4(-2)^3(3)^{-2} = -72$.
c. $\det\left(A^T C^{-1} + (BA)^T\right) = \det(A^T)\det(C^{-1} + B^T) = 0$, since A is singular.

Solution to Question 10. — The standard matrix of $p \rightsquigarrow \begin{pmatrix}p(3)\\p(2)-p(1)\end{pmatrix}$ is $\begin{pmatrix}1 & 3 & 9 & 27\\0 & 1 & 3 & 7\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & 6\\0 & 1 & 3 & 7\end{pmatrix}$, whose null space is generated by $\begin{pmatrix}0\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-6\\-7\\0\\1\end{pmatrix}$; so $-3x + x^2, -6 - 7x + x^3$ is a basis of the subspace in question.

Solution to Question 11. — If A is a 8×7 matrix and the dimension of $\operatorname{Nul}(A^T)$ is 3 then the rank of A is 5 ($= 8 - 3$) and the dimension of $\operatorname{Nul}(A)$ is 2 ($= 7 - 5$).

Solution to Question 12. — If $(AB)^2 = A^2B^2$ and A, B are invertible then $A^{-1}AABBB^{-1} = A^{-1}ABABB^{-1}$, or $AB = BA$.

Solution to Question 13. — Write $\ell\colon \mathbf{p} + t\mathbf{v}$ and $\wp\colon \mathbf{n}^T\mathbf{x} = 4$.
a. $\mathbf{n}^T\mathbf{v} = 0$, so $\mathbf{v}$ is orthogonal to $\mathbf{n}$; therefore, ℓ is parallel to $\wp$.
b. Since ℓ is parallel to $\wp$ and $\mathbf{n}^T\mathbf{p} = 5 \neq 4$, ℓ does not intersect $\wp$.

Solution to Question 14. — Write $\ell_1 : \mathbf{p}_1 + t\mathbf{v}_1$ and $\ell_2 : \mathbf{p}_2 + t\mathbf{v}_2$.
a. The point on ℓ_1 which is closest to the origin is

$$\mathbf{p}_1 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{p}_1) = \begin{pmatrix}3\\1\\-2\end{pmatrix} - \frac{2}{3}\begin{pmatrix}2\\2\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}5\\-1\\-8\end{pmatrix}.$$

b. Since $\mathbf{n} = \frac{1}{3}\mathbf{v}_1 \times \mathbf{v}_2 = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$ is orthogonal to both lines, the distance is

$$\left\|\operatorname{proj}_{\mathbf{n}}(\mathbf{p}_2 - \mathbf{p}_1)\right\| = \frac{|\mathbf{n}^T\mathbf{p}_2 - \mathbf{n}^T\mathbf{p}_1|}{\|\mathbf{n}\|} = \frac{|14 - 9|}{3} = \frac{5}{3}.$$

Solution to Question 15. — Let

$$\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix}1\\2\\1\end{pmatrix}, \qquad \mathbf{v} = \overrightarrow{AC} = \begin{pmatrix}3\\1\\2\end{pmatrix} \qquad \text{and} \qquad \mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix}3\\1\\-5\end{pmatrix}.$$

a. The area of $\triangle ABC$ is $\frac{1}{2}\|\mathbf{n}\| = \frac{1}{2}\sqrt{35}$.

b. The plane which is parallel to $\triangle ABC$ and contains D is defined by the normal equation $\mathbf{n}^T\mathbf{x} = \mathbf{n}^T\overrightarrow{OD}$, or $3x_1 + x_2 - 5x_3 = -21$.

c. The cosine of $\angle BAC$ is $\dfrac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{6}\sqrt{21}$.

Solution to Question 16. — a. Reducing $\begin{pmatrix}2 & -3 & 4 & 7\\1 & -2 & 1 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 5 & 5\\0 & 1 & 2 & 1\end{pmatrix}$ gives $\ell\colon \mathbf{p} + t\mathbf{v}$, where $\mathbf{p} = \begin{pmatrix}5\\1\\0\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}5\\2\\-1\end{pmatrix}$.

b. A normal equation of the plane is $\mathbf{v}^T\mathbf{x} = \mathbf{v}^T\begin{pmatrix}1\\1\\1\end{pmatrix}$, or $5x_1 + 2x_2 - x_3 = 6$, since $\mathbf{v}$ is orthogonal to any plane which is orthogonal to both P_1 and P_2.

c. The line is given by $\begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}5\\2\\-1\end{pmatrix}$, since $\mathbf{v}$ is parallel to both P_1 and P_2.

d. The distance is $\dfrac{\left|\mathbf{n}_1{}^T\begin{pmatrix}1\\1\\1\end{pmatrix} - 7\right|}{\|\mathbf{n}_1\|} = \frac{4}{29}\sqrt{29}$, where $\mathbf{n}_1$ is the normal to P_1.

Solution to Question 17. — Reducing

$$\begin{array}{c}\mathrm{C}\\\mathrm{H}\\\mathrm{O}\end{array}\begin{pmatrix}4 & 0 & -1 & 0\\10 & 0 & 0 & -2\\0 & 2 & -2 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -\frac{1}{5}\\0 & 1 & 0 & -\frac{13}{10}\\0 & 0 & 1 & -\frac{4}{5}\end{pmatrix} \quad \text{gives the generator} \quad \begin{pmatrix}2\\13\\8\\10\end{pmatrix}$$

of its null space. So $2\,C_4H_{10} + 13\,O_2 \to 8\,CO_2 + 10\,H_2O$ is a balancing.

Solution to Question 18. — a. $\det\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = 0 \geqslant 0$ and $\det\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = 0 \geqslant 0$, but $\det\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = -1 \ngeqslant 0$, so S is not closed under addition.

b. If $\det(A) \geqslant 0$ and $\alpha \in \mathbb{R}$ then $\det(\alpha A) = \alpha^2\det(A) \geqslant 0$, so S is closed under scalar multiplication.

Solution to Question 19. — a. The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\dfrac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{-6}{4\sqrt{3}} = -\frac{1}{2}\sqrt{3}$, so this angle is $\frac{5}{6}\pi$ (since it is $\geqslant 0$ and $\leqslant \pi$).

b. $(\mathbf{u}+\mathbf{v})^T(\mathbf{u}+t\mathbf{v}) = \|\mathbf{u}\|^2 + \mathbf{v}^T\mathbf{u} + (\mathbf{u}^T\mathbf{v} + \|\mathbf{v}\|^2)t = 16 - 6 + (3-6)t = 10 - 3t$ is positive if $t < \frac{10}{3}$, in which case the, angle between $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}+t\mathbf{v}$ is $\geqslant 0$ and $< \frac{1}{2}\pi$. The angle between $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}+t\mathbf{v}$ is 0 precisely when $\mathbf{u}+t\mathbf{v}$ is a positive multiple of $\mathbf{u}+\mathbf{v}$, *i.e.*, $t = 1$ (since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent). So the angle between $\mathbf{u}+\mathbf{v}$ is acute if, and only if, $t < \frac{10}{3}$ and $t \neq 1$.